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Belinskaya's theorem is optimal

by

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Abstract. Belinskaya's theorem states that given an ergodic measure-preserving transformation, any other transformation with the same orbits and an L^1 cocycle must be flip-conjugate to it. Our main result shows that this theorem is optimal: for all $p < 1$ the integrability condition on the cocycle cannot be relaxed to being in L^p. This also allows us to answer a question of Kerr and Li: for ergodic measure-preserving transformations, Shannon orbit equivalence does not boil down to flip-conjugacy.

Contents

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1. Introduction. Given two ergodic measure-preserving (invertible) transformations T_1, T_2 of a standard probability space (X, μ), the *conjugacy problem* asks whether there is a third measure-preserving invertible transformation S such that $ST_1 = T_2S$. Although the conjugacy problem is intractable in full generality, various invariants have been devised over the years. Two of the most important ones are the *spectrum* and the *dynamical entropy*. The first completely classifies compact transformations [HvN42], while the second completely classifies Bernoulli shifts [Sin59, Orn70].

In this paper, we are interested in natural weakenings of the conjugacy problem obtained through the notion of orbit equivalence. Two measure-preserving transformations T_1, T_2 are *orbit equivalent* if there is a measure-preserving transformation S such that ST_1S^{-1} and T_2 have the same orbits (such an S is called an orbit equivalence between T_1 and T_2). A stunning theorem of Dye states that all ergodic measure-preserving transformations of a standard probability space are orbit equivalent [Dye59], so orbit equivalence for measure-preserving ergodic transformations is a weakening of conjugacy which turns out to be the trivial relation.

In order to circumvent this indistinguishability, we will compare orbit equivalences between measure-preserving transformations in a quantitative way. This fits into the emerging field of *quantitative orbit equivalence* for group actions. One of its tacit aims is to capture meaningful geometric invariants, such as Følner functions [DK$^+$22], growth rates [Aus16b], etc., or ergodic-theoretic invariants, such as dynamical entropy [Aus16a].

In our setup of measure-preserving transformations, quantifications will be imposed on orbit equivalence cocycles. Given an orbit equivalence S between two ergodic measure-preserving transformations T_1 and T_2, the *orbit equivalence cocycles* $c_1, c_2 : X \to \mathbb{Z}$ are the maps uniquely defined by the following condition: for all $x \in X$,

$$ST_1(x) = T_2^{c_2(x)}S(x) \quad \text{and} \quad T_2S(x) = ST_1^{c_1(x)}(x). \tag{1.1}$$

Belinskaya's theorem is probably the first result on quantitative orbit equivalence. In the literature, it is often stated as a symmetric result on integrable orbit equivalence of ergodic measure-preserving transformations. However, her result is asymmetric and can be stated as follows.

THEOREM 1.1 (Belinskaya [Bel68]). *Let T_1 and T_2 be two ergodic measure-preserving transformations, let S be an orbit equivalence between them and suppose that the previously defined cocycle c_1 is integrable, i.e.*

$$\int_X |c_1(x)|\, d\mu < +\infty.$$

Then T_1 and T_2 are flip-conjugate: either T_1 is conjugate to T_2 or T_1^{-1} is conjugate to T_2.

It is natural to wonder whether Belinskaya's theorem remains valid if one weakens the integrability assumption. For example, one could ask that one of the orbit equivalence cocycles belong to $\mathrm{L}^p(X,\mu)$ for some $p \in (0,1)$.

We will consider more general integrability assumptions. Given a function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, we say that a measurable integer-valued function f is *φ-integrable* if

$$\int_X \varphi(|f(x)|)\, d\mu < +\infty.$$

Our first main result concerns orbit equivalence of measure-preserving transformations for which one of the orbit equivalence cocycles is φ-integrable for some *sublinear* function φ, that is, satisfying $\lim_{t\to+\infty} \varphi(t)/t = 0$. This is for example the case for $\varphi(t) = t^p$ where $p \in (0,1)$. With this integrability condition, the conclusion of Belinskaya's theorem does not hold.

THEOREM 1.2 (see Theorem 4.14). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function and T_1 be an ergodic measure-preserving transformation. Then there is an ergodic measure-preserving transformation T_2 and an orbit equivalence S between T_1 and T_2 such that the associated cocycle c_1 is φ-integrable but the transformations T_1 and T_2 are not flip-conjugate.*

The fact that the hypotheses on φ are fairly weak gives us much freedom. For example, the above theorem even implies that Belinskaya's theorem does not hold if we assume that one of the two orbit equivalence cocycles belongs to $\mathrm{L}^p(X,\mu)$ for *all* $p \in (0,1)$. Indeed, if we consider for instance the sublinear function $\varphi(t) = t/\ln(t+1)$, then φ-integrability implies being in $\mathrm{L}^p(X,\mu)$ for all $p < 1$.

A symmetric way to strengthen Theorem 1.2 involves the concept of φ-integrable orbit equivalence. We say that two measure-preserving transformations are *φ-integrably orbit equivalent* if there is an orbit equivalence S such that *both* orbit equivalence cocycles c_1 and c_2 are φ-integrable. In this context, we obtain a similar conclusion to Theorem 1.2, but we have to make an additional assumption on T_1.

THEOREM 1.3 (see Corollary 3.11). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Let T_1 be an ergodic measure-preserving transformation and assume that $(T_1)^n$ is ergodic for some $n \geq 2$. Then there is another ergodic measure-preserving transformation T_2 such that T_1 and T_2 are φ-integrably orbit equivalent but not flip-conjugate.*

Concrete examples of transformations to which this theorem applies are Bernoulli shifts, irrational rotations on the circle and the m-odometer for any integer m. One can show that the only ergodic measure-preserving transformations that are *not* covered by this theorem are the ones that factor onto the universal odometer, that is, the transformation $t \mapsto t+1$ on the profinite completion $\widehat{\mathbb{Z}}$.

Let us point out that the proof of Theorem 1.2 uses Theorem 1.3, so the two results are not independent. As we will explain later, Theorem 1.2 also depends on the Baire category theorem.

However, Theorem 1.3 is somewhat more explicit. The starting point of the proof is the following simple construction, which was already used in [LM18, Thm. 4.8]. Let us fix an ergodic transformation T_1 with $(T_1)^n$ ergodic. Suppose we have a periodic transformation P all of whose orbits have cardinality n and are contained in those of T_1. Consider the transformation T_2 obtained by composing P with the transformation induced by T_1 on a fundamental domain of P. Then T_1 and T_2 have the same orbits. However, $(T_2)^n$ is not ergodic and thus T_1 and T_2 are not flip-conjugate. The heart of our proof is therefore to construct P so that the orbit equivalence cocycles between T_1 and T_2 satisfy the required integrability conditions.

For many concrete measure-preserving transformations, explicit examples of such periodic transformations P with specific integrability conditions can be obtained. We will give details in the case of the Bernoulli shift; see Example 3.3.

Shannon orbit equivalence and dynamical entropy. A remarkable consequence of Theorem 1.3 can be stated in the context of *Shannon orbit equivalence*, as defined by Kerr and Li [KL21]. Two measure-preserving transformations are Shannon orbit equivalent if there exists an orbit equivalence between them whose orbit equivalence cocycles c_1 and c_2 both have finite Shannon entropy. In [Aus16a], it was implicitly shown that, among actions of finitely generated amenable groups which are not virtually cyclic, Shannon orbit equivalence preserves dynamical entropy. This was then generalized in [KL21] to countable amenable groups that are neither locally finite nor virtually cyclic, as well as to some nonamenable groups. Kerr and Li implicitly asked whether dynamical entropy is an invariant of Shannon orbit equivalence for measure-preserving transformations and wondered whether Shannon orbit equivalence could actually directly imply flip-conjugacy. They answered the first part of this question positively in [KL22]. Here, we show that Shannon orbit equivalence does not boil down to flip-conjugacy.

THEOREM 1.4 (see Theorem 3.18). *Let $T_1 \in \mathrm{Aut}(X,\mu)$ be an ergodic transformation and assume that $(T_1)^n$ is ergodic for some $n \geq 2$. Then there exists $T_2 \in \mathrm{Aut}(X,\mu)$ such that T_1 and T_2 are Shannon orbit equivalent but not flip-conjugate.*

The above theorem is obtained by applying Theorem 1.3 with any sublinear function φ such that $\ln(1+t) = O(\varphi(t))$. Indeed, for any such function,

φ-integrable orbit equivalence implies Shannon orbit equivalence; see Theorem 3.16.

We also observe that Shannon orbit equivalence preserves finiteness of dynamical entropy; see Proposition 3.21. This is now subsumed by a recent preprint of Kerr and Li, who proved that the dynamical entropy is preserved under Shannon orbit equivalence [KL22].

QUESTION 1.5. For which unbounded sublinear metric-compatible functions φ is it true that dynamical entropy is an invariant of φ-integrable orbit equivalence?

By the above discussion, we already know that this holds for all φ such that $\ln(1+t) = O(\varphi(t))$. On the other hand, using Dye's theorem, it is not hard to see that any two ergodic measure-preserving transformations are φ-integrably orbit equivalent for *some* sublinear unbounded function φ (cf. [DK$^+$22, proof of Prop. 4.24]). So not every sublinear unbounded function satisfies the condition of the question.

φ-integrable full groups. The proof of both our main results will make crucial use of the notion of *φ-integrable full group*. Whenever T is an ergodic measure-preserving transformation of the probability space (X, μ), Dye defined a Polish group $[T]$, called the *full group* of T. This group is by definition the set of all measure-preserving transformations U of (X, μ) whose orbits are contained in T-orbits. More precisely, $U \in [T]$ if there is a function c_U, called the *T-cocycle* of U, such that $U(x) = T^{c_U(x)}(x)$ for all $x \in X$. The above stated theorem of Dye, that all ergodic transformations are orbit equivalent, was originally stated in terms of full groups: whenever T_1 and T_2 are ergodic transformations, the full groups $[T_1]$ and $[T_2]$ are conjugate.

In our context, once φ is fixed, the reasonable analogue of the full group associated to this integrability condition would be the set of transformations $U \in [T]$ such that the cocycle c_U is φ-integrable. However, for this set to be a subgroup of $[T]$, we will have to impose a mild restriction on φ. We say that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a *metric-compatible function* if

- (subadditivity) for all $s, t \in \mathbb{R}_+$, $\varphi(s+t) \le \varphi(s) + \varphi(t)$;
- (separation) $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$;
- (monotonicity) φ is non-decreasing.

The name "metric-compatible" comes from the observation that whenever d is a metric and φ a metric-compatible function, then $\varphi \circ d$ is also a metric. The following theorem is a combination of Lemma 2.14 and Theorem 4.1.

THEOREM 1.6. *Let φ be a metric-compatible function and let T be a measure-preserving transformation of the probability space (X, μ). Then the set*

$$[T]_\varphi := \Big\{U \in [T] \colon \int_X \varphi(|c_U(x)|)\, d\mu < +\infty\Big\}$$

is a group. Moreover, the function

$$\mathsf{d}_{\varphi,T}(U,V) := \int_X \varphi(|c_U(x) - c_V(x)|)\, d\mu$$

is a complete, right-invariant and separable metric on $[T]_\varphi$ whose induced topology is a group topology. In particular, $([T]_\varphi, \mathsf{d}_{\varphi,T})$ is a Polish group.

It turns out that any sublinear function is dominated by a sublinear metric-compatible function; see Lemma 2.12. This will allow us to reduce the proof of Theorems 1.2 and 1.3 to the case where φ is metric-compatible, and thereby to exploit the group structure of $[T]_\varphi$.

Genericity of weak mixing. Let us come back to Theorem 1.2. As the conclusions of Theorem 1.3 are stronger, we just need to show Theorem 1.2 whenever $(T_1)^n$ is non-ergodic for all $n \geq 2$. Observe that this condition is incompatible with the notion of *weak mixing*, as all the powers of any weakly mixing transformation are ergodic. Therefore our strategy is to provide for every ergodic transformation T_1 a weakly mixing transformation T_2 which has the same orbits as T_1 and whose T_1-cocycle is φ-integrable. We do not have any constructive argument for this and we proceed through the Baire category theorem, inspired by a similar result in the context of full groups of ergodic measure-preserving equivalence relations (see [Kec10, Thm. 3.6]).

THEOREM 1.7 (see Theorem 4.15). *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic element. Then the set of all measure-preserving transformations in $[T]_\varphi$ which are weakly mixing and have the same orbits as T is a dense G_δ set in the Polish space of aperiodic transformations of $[T]_\varphi$.*

Besides the Baire category theorem, there are two other main ingredients in the proof of Theorem 1.7. One is a result of Conze [Con72] which claims that starting from any ergodic mesure-preserving transformation, the first return map to a *generic* measurable subset gives rise to a weakly mixing transformation. The second is a sublinear ergodic theorem which may be of independent interest.

THEOREM 1.8 (see Theorem 4.5). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $U \in \mathrm{Aut}(X,\mu)$ and let $f \colon X \to \mathbb{C}$ be measurable such that $\varphi(|f|) \in \mathrm{L}^1(X,\mu)$. Then for almost every $x \in X$,*

$$\lim_n \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|\Big) = 0.$$

The convergence also holds in L^1, *that is,*

$$\lim_n \int_X \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|\Big)\, d\mu = 0.$$

Outline of the paper. In Section 2, after a few preliminaries, we present the framework and establish basic properties of φ-integrable full groups. In Section 3, we explain our construction of periodic transformations in φ-integrable full groups and use it to prove Theorem 1.2. In Section 4, we first prove that φ-integrable full groups are Polish groups. We then use the Baire category theorem and prove that weakly mixing elements are generic in the set of aperiodic elements in $[T]_\varphi$. Combining this with Theorem 1.2, we finally prove Theorem 1.3. In the Appendix, we also present a proof of Belinskaya's theorem which is due to Katznelson and is not publicly available to our knowledge.

2. Quantitative orbit equivalence and full groups

2.1. Preliminaries. Throughout the paper, (X,μ) will denote a standard probability space without atoms. Recall that such spaces are measure isomorphic to the interval $[0,1]$ equipped with the Lebesgue measure. A bimeasurable bijection $T\colon X\to X$ is a *measure-preserving transformation* of (X,μ) if for all measurable sets $A\subseteq X$, one has $\mu(T^{-1}(A))=\mu(A)$. We denote by $\mathrm{Aut}(X,\mu)$ the group of all measure-preserving transformations of (X,μ), two such transformations being identified if they coincide on a conull set. The group $\mathrm{Aut}(X,\mu)$ will be equipped with the *uniform metric* d_u defined by

$$d_u(T_1,T_2) := \mu(\{x\in X\colon T_1(x)\neq T_2(x)\}).$$

This metric is bi-invariant and complete [Hal17, p. 73].

Remark 2.1. We will often implicitly use the fact that two measure-preserving transformations which belong to the same class in $\mathrm{Aut}(X,\mu)$ agree on an invariant conull set.

The *support* of a measure-preserving transformation $T\in\mathrm{Aut}(X,\mu)$ is the measurable set $\mathrm{supp}(T) := \{x\in X\colon T(x)\neq x\}$.

A measure-preserving transformation $T\in\mathrm{Aut}(X,\mu)$ is *periodic* if the T-orbit of almost every $x\in X$ is finite. A *fundamental domain* of a periodic transformation $T\in\mathrm{Aut}(X,\mu)$ is a measurable subset $D\subseteq X$ which intersects almost every T-orbit at exactly one point. Every periodic transformation admits such a fundamental domain, as can be seen by fixing a Borel linear order $<$ on X and taking for D the set of $<$-least points in each orbit of the transformation. A measure-preserving transformation $T\in\mathrm{Aut}(X,\mu)$

is *aperiodic* if the T-orbit of almost every $x \in X$ is infinite. It is *ergodic* if every T-invariant measurable set is either null or conull.

The *full group* of a measure-preserving transformation T is the group

$$[T] := \{U \in \mathrm{Aut}(X, \mu) \colon \text{for a.e. } x \in X,\ \exists n \in \mathbb{Z},\ U(x) = T^n(x)\}.$$

REMARK 2.2. By Remark 2.1, $U \in [T]$ if and only if, for a.e. $x \in X$, the U-orbit of x is contained in the T-orbit of x.

Two measure-preserving transformations $T_1, T_2 \in \mathrm{Aut}(X, \mu)$ *have the same orbits* if, for almost every $x \in X$, the T_1-orbit of x coincides with the T_2-orbit of x. By the above remark, this is equivalent to the following condition: $T_1 \in [T_2]$ and $T_2 \in [T_1]$. We say that two measure-preserving transformations $T_1, T_2 \in \mathrm{Aut}(X, \mu)$ are *orbit equivalent* if there exists $S \in \mathrm{Aut}(X, \mu)$ such that ST_1S^{-1} and T_2 have the same orbits, that is, $ST_1S^{-1} \in [T_2]$ and $T_2 \in [ST_1S^{-1}]$.

Fix an aperiodic transformation $T \in \mathrm{Aut}(X, \mu)$. Any $U \in [T]$ is completely determined by its T*-cocycle*, defined as the unique function $c_U \colon X \to \mathbb{Z}$ satisfying the equation $U(x) = T^{c_U(x)}(x)$ for all $x \in X$. The T-cocycle satisfies the so-called *cocycle identity*: given $U, V \in [T]$, we have

$$c_{UV}(x) = c_U(V(x)) + c_V(x) \quad \text{for all } x \in X. \tag{2.1}$$

Let $T \in \mathrm{Aut}(X, \mu)$ and $A \subseteq X$ be a measurable subset. The *first return time* of T to A is the map $n_{T,A} \colon A \to \mathbb{N}^*$ defined by

$$n_{T,A}(x) := \min\{n \in \mathbb{N}^* \colon T^n(x) \in A\}.$$

This function is well-defined up to measure zero by Poincaré's recurrence theorem. For convenience, we extend $n_{T,A}$ to all X, setting it to be 0 on $X \setminus A$. Kac's lemma [Kac47] yields the inequality

$$\int_X n_{T,A}(x)\, d\mu \le 1. \tag{2.2}$$

The *first return map* of T with respect to A is the transformation $T_A \in [T] \le \mathrm{Aut}(X, \mu)$ defined by

$$T_A(x) := T^{n_{T,A}(x)}(x).$$

By definition, we have $\mathrm{supp}(T_A) = A$, and $x, y \in A$ are in the same T-orbit if and only if they are in the same T_A-orbit. Whenever T is aperiodic, the first return time $n_{T,A}$ coincides with the T-cocycle c_{T_A} of T_A.

LEMMA 2.3. *Let $T \in \mathrm{Aut}(X, \mu)$, and let $P \in \mathrm{Aut}(X, \mu)$ be a periodic transformation and D a fundamental domain of P. Let $U := T_D P$. Then the following are true:*

(i) $U_D = T_D$.
(ii) *If $x \in D$ and $n(x)$ is the cardinality of the P-orbit of x, then $n_{U,D}(x) = n(x)$.*
(iii) *If $P \in [T]$, then T and U have the same orbits.*

Proof. We first prove (i) and (ii). Clearly $U_D(x) = T_D(x) = x$ for every $x \notin D$. Since D is a fundamental domain for P, for all $x \in D$ and $i \in \{1, \dots, n(x)-1\}$ we have $P^i(x) \notin D$. Since $T_D(x) = x$ for all $x \notin D$, we deduce by induction that

$$U^i(x) = P^i(x) \notin D \quad \text{ for all } x \in D \text{ and } i \in \{1, \dots, n(x)-1\}.$$

So for all $x \in D$, one has $U^{n(x)}(x) = UU^{n(x)-1}(x) = T_D P^{n(x)}(x) = T_D(x)$. This shows (i) and (ii).

We now prove (iii). Clearly, $U \in [T]$. We need to show that $T \in [U]$. Observe that for almost every x, the U-orbit of x meets D: indeed, if $x \in P^i(D)$ for $i \in \{1, \dots, n(x)-1\}$, then $U^{-i}(x) = P^{-i}(x) \in D$. Since being in the same orbit is an equivalence relation, it is enough to show that any two points in D which belong to the same T-orbit are in the same U-orbit. This follows directly from (i). ∎

We will also need the following lemma, which can be proven with the same kind of arguments as above.

LEMMA 2.4. *Let $U \in \mathrm{Aut}(X, \mu)$ and let A be a measurable subset of X which intersects every U-orbit. Then $(U_A)^{-1}U$ is periodic and A is a fundamental domain for it.*

Proof. Since A intersects every U-orbit, for almost every $x \in X \setminus A$ there exists a smallest $n \geq 1$ such that $U^{-n}(x) \in A$. Remark that $((U_A)^{-1}U)^{-n}(x) = U^{-n}(x) \in A$ and hence A intersects every $(U_A)^{-1}U$-orbit. If $x \in A$, then for every $0 \leq n < n_{U,A}(x)$ we have

$$((U_A)^{-1}U)^n(x) = U^n(x) \notin A,$$
$$((U_A)^{-1}U)^{n_{U,A}(x)}(x) = U_A^{-1} U U^{n_{U,A}(x)-1}(x) = x.$$

Since A intersects every $(U_A)^{-1}U$-orbit, we find that every $(U_A)^{-1}U$-orbit is finite and A is a fundamental domain for $(U_A)^{-1}U$. ∎

2.2. φ-integrable orbit equivalence and full groups. We first define the notion of φ-integrable orbit equivalence.

DEFINITION 2.5. Fix $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$. Two aperiodic transformations $T_1, T_2 \in \mathrm{Aut}(X, \mu)$ are *φ-integrably orbit equivalent* if there exists $S \in \mathrm{Aut}(X, \mu)$ such that ST_1S^{-1} and T_2 have the same orbits and their respective cocycles are φ-integrable. To be more precise, we ask that

$$\int_X \varphi(|c_{ST_1S^{-1}}(x)|)d\mu < +\infty \quad \text{and} \quad \int_X \varphi(|c_{T_2}(x)|)d\mu < +\infty,$$

where $c_{ST_1S^{-1}}$ is the T_2-cocycle of ST_1S^{-1} and c_{T_2} is the ST_1S^{-1}-cocycle of T_2, defined for all $x \in X$ by the equations

$$ST_1S^{-1}(x) = T_2^{c_{ST_1S^{-1}}(x)}(x) \quad \text{and} \quad T_2(x) = (ST_1S^{-1})^{c_{T_2}(x)}(x).$$

When $\varphi(t) = t^p$ for some $p \in (0, +\infty)$, we recover the notion of L^p orbit equivalence.

REMARK 2.6. We warn the reader that even though the term "L^p orbit equivalence" is often used in the literature, this terminology may sound a bit deceptive. Indeed, since the integrability condition has no reason to be preserved under composition of orbit equivalences, we do not expect φ-integrable (even L^p) orbit equivalence to be an equivalence relation for every concave function φ, although we do not have any counterexample. The fact that it is the case for $p = 1$ seems to be a rather artificial consequence of Belinskaya's theorem.

In our work, the function φ is at most linear and for our main theorems the function is assumed to be *sublinear*, that is, $\lim_{t\to+\infty} \varphi(t)/t = 0$. For example we are interested in the case of L^p orbit equivalence for $p \leq 1$, or in the case of $\varphi(t) = \log(1 + t)$.

In the context of φ-orbit equivalence, it is natural to consider the set of measure-preserving transformations U whose cocycle c_U is φ-integrable. In order for this set to be a group, the following conditions on φ are required.

DEFINITION 2.7. A function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is *metric-compatible* if

- (subadditivity) for all $s, t \in \mathbb{R}_+$, $\varphi(s + t) \leq \varphi(s) + \varphi(t)$;
- (separation) $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$;
- (monotonicity) φ is non-decreasing.

EXAMPLE 2.8. Any concave function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ is metric-compatible. In particular, for every $p \leq 1$, the function $\varphi(t) = t^p$ is metric-compatible. It is moreover sublinear whenever $p < 1$. Other examples of sublinear metric-compatible functions are $\varphi(t) = \log(1 + t)$ and $\varphi(t) = t/\log(2 + t)$.

The term "metric-compatible" was coined because of the following property: whenever d is a metric on a set X, so is $\varphi \circ d$.

CONVENTION. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. For all $t \in \mathbb{R}$, we use the notation

$$|t|_\varphi := \varphi(|t|).$$

The map $(s, t) \mapsto |s - t|_\varphi$ is a metric on $\mathbb{R}$.

DEFINITION 2.9. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. The *φ-integrable full group* of an aperiodic transformation $T \in \mathrm{Aut}(X, \mu)$ is

$$[T]_\varphi := \Big\{U \in [T]\colon \int_X |c_U(x)|_\varphi \, d\mu < +\infty\Big\},$$

where $c_U\colon X \to \mathbb{Z}$ denotes the T-cocycle of U.

Given a metric-compatible function φ, the φ-integrable full group $[T]_\varphi$ is indeed a group: given $U, V \in [T]_\varphi$, the cocycle identity implies that

$$c_{UV^{-1}}(x) = c_U(V^{-1}(x)) + c_{V^{-1}}(x) = c_U(V^{-1}(x)) - c_V(V^{-1}(x)).$$

We then get

$$\begin{aligned}\int_X |c_{UV^{-1}}(x)|_\varphi \, d\mu &\le \int_X |c_U(V^{-1}(x))|_\varphi \, d\mu + \int_X |c_V(V^{-1}(x))|_\varphi \, d\mu \\ &= \int_X |c_U(x)|_\varphi \, d\mu + \int_X |c_V(x)|_\varphi \, d\mu < +\infty.\end{aligned} \tag{2.3}$$

Example 2.10. If φ is any bounded metric-compatible function, then $[T]_\varphi = [T]$, and if φ is the identity map, then we recover the L^1 full group $[T]_1$ defined by the third named author [LM18]. Any other such φ gives rise to new ([1]) examples of full groups, such as L^p full groups $[T]_p$ for $0 < p < 1$ obtained with the function $\varphi(t) = t^p$, or else $[T]_{\log}$ obtained with the function $\varphi(t) = \log(1+t)$.

Remark 2.11. Given a metric-compatible function φ, it is now straightforward to check that two aperiodic transformations $T_1, T_2 \in \mathrm{Aut}(X,\mu)$ are φ-integrably orbit equivalent if and only if there is $S \in \mathrm{Aut}(X,\mu)$ such that $ST_1S^{-1} \in [T_2]_\varphi$ and $T_2 \in [ST_1S^{-1}]_\varphi$. However, the notion of φ-orbit equivalence is a priori weaker than conjugacy of φ-integrable full groups. Indeed, conjugacy of φ-integrable full groups is an equivalence relation, but φ-integrable orbit equivalence may not be; see Remark 2.6. This is in contrast with the case of classical orbit equivalence; see [Kec10, Thm. 4.1].

In our two main results, namely Theorems 1.2 and 1.3, the sublinear function φ is not assumed to be metric-compatible. The following lemma will allow us to reduce to the case where φ is in addition metric-compatible.

Lemma 2.12. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Then there is a sublinear metric-compatible function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \le \psi(t)$ for all t large enough.*

Proof. Set

$$\theta\colon \mathbb{R}_+^* \to \mathbb{R}_+, \qquad \theta(t) := \min\left(1, \sup_{s \ge t} \frac{\varphi(s)+1}{s}\right),$$

$$\psi\colon \mathbb{R}_+ \to \mathbb{R}_+, \qquad \psi(t) := \int_0^t \theta(s)\, ds.$$

Noting that θ is positive-valued and non-increasing, it is straightforward to check that ψ is non-decreasing, subadditive and that $\psi(t) = 0$ if and only

([1]) We can actually characterize when $[T]_\varphi = [T]_\psi$ and more generally when $[T]_\varphi \le [T]_\psi$; see Proposition 4.2.

if $t=0$. Moreover, the fact that $\theta(t)$ tends to 0 as t approaches $+\infty$ implies that ψ is sublinear. Now note that for every $t\in\mathbb{R}_+^*$,

$$\psi(t)=\int_0^t \theta(s)\,ds \geq \int_0^t \theta(t)\,ds = t\theta(t).$$

For $t\in\mathbb{R}_+^*$ large enough so that $\sup_{s\geq t}\frac{\varphi(s)+1}{s}\leq 1$ we finally have

$$t\theta(t)=t\sup_{s\geq t}\frac{\varphi(s)+1}{s}=t\sup_{s\geq t}\frac{\varphi(s)+1}{s}\geq t\frac{\varphi(t)+1}{t}=\varphi(t)+1\geq\varphi(t),$$

so we are done. ∎

Remark 2.13. Given a sublinear function φ, Lemma 2.12 grants us a sublinear metric-compatible function ψ such that $\varphi(t)\leq\psi(t)$ for all t large enough. Therefore, for any measurable function $f\colon X\to\mathbb{Z}$ we have

$$\int_X \psi(|f(x)|)\,d\mu<+\infty \quad \text{implies} \quad \int_X \varphi(|f(x)|)\,d\mu<+\infty.$$

In particular, ψ-integrable orbit equivalence implies φ-orbit equivalence, and any element in a ψ-integrable full group will have φ-integrable cocycle.

We will state most of our results in the comfortable context of (sublinear) metric-compatible functions. However, many of our statements could be easily generalized to the general context of sublinear functions through Remark 2.13. We will explicitly do so only in our main theorems, Theorems 1.2 and 1.3.

2.3. Metric properties of φ-integrable full groups. We now introduce and study a natural extended pseudo-metric on full groups from which φ-integrable full groups naturally arise.

Lemma 2.14. *Let $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$ be a metric-compatible function and let $T\in\mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Let $\mathsf{d}_{\varphi,T}\colon[T]\times[T]\to\mathbb{R}_+\cup\{+\infty\}$ be the function defined by*

$$\mathsf{d}_{\varphi,T}(U,V):=\int_X |c_U(x)-c_V(x)|_\varphi\,d\mu.$$

Then the following are true:

(i) *The group $[T]_\varphi$ is determined by $\mathsf{d}_{\varphi,T}$:*

$$[T]_\varphi=\{U\in[T]\colon \mathsf{d}_{\varphi,T}(U,\mathrm{id})<+\infty\}.$$

(ii) *The restriction of $\mathsf{d}_{\varphi,T}$ to $[T]_\varphi\times[T]_\varphi$ is a metric on $[T]_\varphi$ which is right-invariant, that is, for all $U,V,W\in[T]_\varphi$,*

$$\mathsf{d}_{\varphi,T}(UW,VW)=\mathsf{d}_{\varphi,T}(U,V).$$

Proof. Item (i) is an immediate consequence of the definition of $[T]_\varphi$.

Let us now prove (ii). The fact that $\mathsf{d}_{\varphi,T}$ is a metric is a straightforward consequence of the fact that $(s,t) \mapsto |s-t|_\varphi$ is a metric on $\mathbb{R}$. The right-invariance follows from the cocycle identity (2.1) and the fact that the transformation W is measure-preserving:

$$\begin{aligned}\mathsf{d}_{\varphi,T}(UW,VW) &= \int_X |c_{UW}(x) - c_{VW}(x)|_\varphi \, d\mu \\ &= \int_X |c_U(W(x)) - c_V(W(x))|_\varphi \, d\mu \\ &= \int_X |c_U(x) - c_V(x)| \, d\mu = \mathsf{d}_{\varphi,T}(U,V). \quad \blacksquare\end{aligned}$$

Example 2.15. Consider the metric-compatible function

$$\varphi := \min(\mathrm{id}_{\mathbb{R}_+}, 1).$$

Then it is straightforward to check that $\mathsf{d}_{\varphi,T} = d_u$ is the uniform metric on $[T] = [T]_\varphi$.

Another example is obtained by taking $\varphi := \mathrm{id}_{\mathbb{R}_+}$; we then recover the L^1 metric on the L^1 full group $[T]_1 = [T]_\varphi$.

In order to compare φ-integrable full groups, we are led to compare asymptotically metric-compatible functions. We will use the following standard notation: given two real-valued functions f and g, we write $f(t) = O(g(t))$ as $t \to +\infty$ if there exist $t_0 > 0$ and $C > 0$ such that for all $t > t_0$, we have $|f(t)| \leq C|g(t)|$. Since the functions we consider are subadditive, it is enough to compare them on the integers.

Lemma 2.16. *Let φ and ψ be metric compatible functions. Then the following are equivalent:*

(i) *$\varphi(t) = O(\psi(t))$ as $t \to +\infty$.*
(ii) *There exists $C > 0$ such that $\varphi(t) \leq C\psi(t)$ for all $t \geq 1$.*
(iii) *There exists $C > 0$ such that $\varphi(k) \leq C\psi(k)$ for all $k \in \mathbb{N}$.*

Proof. We first prove that (i) implies (ii). Let $t_0 \geq 1$ and $D > 0$ be such that for all $t > t_0$, we have $\varphi(t) \leq D\psi(t)$. Set $C := \max(D, \varphi(t_0)/\psi(1))$ and observe that since φ and ψ are non-decreasing, $\varphi(t) \leq C\psi(t)$ for all $t \geq 1$.

The implication (ii)$\Rightarrow$(iii) is straightforward, so it remains to prove (iii)$\Rightarrow$(i). Let $C > 0$ be such that $\varphi(k) \leq C\psi(k)$ for all integers $k \in \mathbb{N}$. Fix a real number $t \geq 2$ and let $n \in \mathbb{N}^*$ be such that $n \leq t < n+1$. Then

$$\varphi(t) \leq \varphi(n+1) \leq C\psi(n+1) \leq C(\psi(t) + \psi(1)) \leq C\left(1 + \frac{\psi(1)}{\psi(t)}\right)\psi(t),$$

and since $\psi(t) \geq \psi(1)$ for every $t \geq 1$, the proof is complete. $\blacksquare$

We now compare φ-integrable full groups for different metric-compatible functions.

LEMMA 2.17. *Let φ and ψ be metric-compatible functions and fix an aperiodic transformation $T \in \mathrm{Aut}(X,\mu)$. If $\varphi(t) = O(\psi(t))$ as $t \to +\infty$, then $[T]_\psi \leq [T]_\varphi$. Moreover, the inclusion map is Lipschitz.*

Proof. By Lemma 2.16, there is $C > 0$ such that $\varphi(k) \leq C\psi(k)$ for all $k \in \mathbb{N}$. Let $U, V \in [T]$. Then for almost every $x \in X$,

$$|c_U(x) - c_V(x)|_\varphi \leq C|c_U(x) - c_V(x)|_\psi.$$

Integrating over X, we get $\mathsf{d}_{\varphi,T}(U,V) \leq C\mathsf{d}_{\psi,T}(U,V)$. The lemma now follows immediately. ∎

COROLLARY 2.18. *Whenever T is an aperiodic measure-preserving transformation, we have*

$$[T]_1 \leq [T]_\varphi \leq [T],$$

and the inclusion maps are Lipschitz.

Proof. Since φ is subadditive, we have $\varphi(t) = O(t)$ as $t \to +\infty$. Moreover, $\min(1,t) = O(\varphi(t))$ as $t \to +\infty$. The conclusion now follows from Lemma 2.17. ∎

We will show in Proposition 4.2 that the implication in Lemma 2.17 is an equivalence. For this, we will make a crucial use of the fact that the topologies induced by these metrics are Polish group topologies; see Theorem 4.1.

REMARK 2.19. Let $d_T \colon X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ be the extended metric on X defined by

$$d_T(x,y) := \inf\{n \in \mathbb{N} \colon T^n(x) = y \text{ or } T^n(y) = x\}.$$

Then by definition of the T-cocycle of any $U \in [T]$, we know that for all $x \in X$, $d_T(U(x),x) = |c_U(x)|$. For all $U, V \in [T]_\varphi$, the cocycle identity implies that $c_{UV^{-1}}(x) = c_U(V^{-1}(x)) - c_V(V^{-1}(x))$. Since V preserves the measure, we obtain

$$\begin{aligned}
\mathsf{d}_{\varphi,T}(U,V) &= \int_X |c_U(V^{-1}(x)) - c_V(V^{-1}(x))|_\varphi \, d\mu \\
&= \int_X |c_{UV^{-1}}(x)|_\varphi \, d\mu = \int_X \varphi(d_T(UV^{-1}(x),x)) \, d\mu \\
&= \int_X \varphi(d_T(U(x),V(x))) \, d\mu.
\end{aligned}$$

We will not use this formula hereafter. However, this point of view allows one to define φ-integrable full groups of not necessarily free actions of finitely generated groups. Some of the arguments given in the present paper work in this wider context; this will be examined in an upcoming work.

3. Flexibility of φ-integrable orbit equivalence

3.1. Construction of cycles in φ-integrable full groups. An *n-cycle*, $n \geq 2$, is a periodic transformation $P \in \mathrm{Aut}(X, \mu)$ whose orbits have cardinality either 1 or n. The aim of this section is to prove the following result.

THEOREM 3.1. *Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $T \in \mathrm{Aut}(X, \mu)$ be aperiodic. Then for all measurable $A \subseteq X$ and all integers $n \geq 2$, there exists an n-cycle $P \in [T]_\varphi$ whose support is A.*

REMARK 3.2. The hypothesis that φ is sublinear is necessary, as the result is false for L^1 full groups of certain aperiodic transformations. Indeed, if $T \in \mathrm{Aut}(X, \mu)$ is ergodic, then there exists an n-cycle in $[T]_1$ whose support is $A \subseteq X$ if and only if $\exp(2i\pi/n)$ is in the spectrum of the restriction of T_A to A [LM18, Thm. 4.8]. In particular, the L^1 full group of the Bernoulli shift contains no n-cycle with full support for any $n \geq 2$. By contrast, Theorem 3.1 says that as long as $p < 1$, its L^p full group contains an n-cycle of full support for every $n \geq 2$.

EXAMPLE 3.3. In certain concrete situations, we can exhibit explicit involutions. Let T be the Bernoulli shift on $(\{0,1\}, \kappa)^{\otimes \mathbb{Z}}$, where κ is the uniform measure on $\{0,1\}$. Then for every $0 < p < 1/2$, there exists an involution in $[T]_p$ with full support and fundamental domain $X_0 := \{(x_n)_{n \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} \colon x_0 = 0\}$.

Indeed, for all $x \in X_0$, let $N(x)$ be the infimum of $n \geq 1$ such that 1 appears strictly more often than 0 in $\{x_1, \ldots, x_n\}$. Then the map $\pi \colon X_0 \ni x \mapsto T^{N(x)}(x) \in \{0,1\}^{\mathbb{Z}} \setminus X_0$ is almost everywhere well-defined and injective. Thus it can be extended to an involution $P \in [T]$ with full support and fundamental domain X_0. Standard estimates on the first return time to 0 of the simple random walk on $\mathbb{Z}$ [Fel68, Chap. III.2] imply that P belongs to $[T]_p$ for all $0 < p < 1/2$.

REMARK 3.4. Theorem 3.1 tells us that any measurable subset $A \subseteq X$ is the support of an involution. The situation is less flexible regarding fundamental domains. For example, the subset X_0 introduced in the previous example cannot be the fundamental domain of any involution in the L^p full group of the Bernoulli shift for $1/2 \leq p \leq 1$, as a consequence of a result of Liggett [Lig02]. Note that his result is more general and stated in probabilistic terms; the connection to our context and a purely ergodic-theoretic version of his proof will be presented in the second named author's PhD thesis [Jos22, Chap. 3].

A *partial measure-preserving transformation* of (X, μ) is a bimeasurable measure-preserving bijection π between two measurable subsets $\mathrm{dom}(\pi)$ and

$\mathrm{rng}(\pi)$ of X, called respectively the *domain* and the *range* of π. The *support* of π is the set

$$\mathrm{supp}(\pi) := \{x \in \mathrm{dom}(\pi) \colon \pi(x) \neq x\} \cup \{x \in \mathrm{rng}(\pi) \colon \pi^{-1}(x) \neq x\}.$$

A *pre-cycle* of length $n \geq 2$ is a partial measure-preserving transformation $\pi \colon \mathrm{dom}(\pi) \to \mathrm{rng}(\pi)$ of (X, μ) such that if we set $B := \mathrm{dom}(\pi) \setminus \mathrm{rng}(\pi)$, then

- $\{\pi^0(B), \dots, \pi^{n-2}(B)\}$ is a partition of $\mathrm{dom}(\pi)$,
- $\{\pi^1(B), \dots, \pi^{n-1}(B)\}$ is a partition of $\mathrm{rng}(\pi)$.

The set $B = \mathrm{dom}(\pi) \setminus \mathrm{rng}(\pi)$ is called the *basis* of the pre-cycle π.

A pre-cycle π of length n can be extended to an n-cycle P called the *closing cycle* of π, defined as follows:

$$P(x) := \begin{cases} \pi(x) & \text{if } x \in \mathrm{dom}(\pi), \\ \pi^{-(n-1)}(x) & \text{if } x \in \mathrm{rng}(\pi) \setminus \mathrm{dom}(\pi), \\ x & \text{else.} \end{cases}$$

Observe that the support of P coincides with the support of the pre-cycle π and that the basis B is a fundamental domain for the restriction of P to its support. A pre-cycle π is *induced* by $T \in \mathrm{Aut}(X, \mu)$ if for all $x \in \mathrm{dom}(\pi)$, we have $\pi(x) = T_{\mathrm{supp}(\pi)}(x)$.

LEMMA 3.5. *Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. Let $T \in \mathrm{Aut}(X, \mu)$ be an aperiodic transformation, let π be a pre-cycle induced by T and let P be its closing cycle. Then*

$$\mathsf{d}_{\varphi,T}(P, \mathrm{id}) \leq 2\mathsf{d}_{\varphi,T}(T_{\mathrm{supp}(\pi)}, \mathrm{id}).$$

In particular, P belongs to $[T]_\varphi$.

Proof. Let n be the length of the pre-cycle π, let $A := \mathrm{supp}(\pi)$ and let $B := \mathrm{dom}(\pi) \setminus \mathrm{rng}(\pi)$ the basis of π. Since π is induced by T, for all $x \in \mathrm{dom}(\pi)$ one has $\pi(x) = P(x) = T_A(x)$. This implies that $c_P(x) = c_{T_A}(x)$ for all $x \in \mathrm{dom}(\pi)$. Thus,

$$\begin{aligned} \mathsf{d}_{\varphi,T}(P, \mathrm{id}) &= \int\limits_{\mathrm{dom}(\pi)} |c_{T_A}(x)|_\varphi \, d\mu + \int\limits_{P^{n-1}(B)} |c_{P^{-(n-1)}}(x)|_\varphi \, d\mu \\ &\leq \mathsf{d}_{\varphi,T}(T_A, \mathrm{id}) + \int\limits_B |c_{P^{n-1}}(x)|_\varphi \, d\mu. \end{aligned}$$

Moreover, for all $x \in B$, the cocycle identity yields

$$|c_{P^{n-1}}(x)|_\varphi \leq |c_P(x)|_\varphi + |c_P(P(x))|_\varphi + \cdots + |c_P(P^{n-2}(x))|_\varphi.$$

We now use the fact that P preserves the measure and that $\mathrm{dom}(\pi) =$

$B \sqcup P(B) \sqcup \cdots \sqcup P^{n-2}(B)$ to get

$$\int_B |c_{P^{n-1}}(x)|_\varphi \, d\mu \le \int_{\mathrm{dom}(\pi)} |c_P(x)|_\varphi \, d\mu \le \int_X |c_{T_A}(x)|_\varphi \, d\mu,$$

which concludes the proof. ∎

Kac's lemma, that is, equation (2.2), implies that for every measurable $A \subseteq X$, the first return map T_A belongs to $[T]_1$, which is contained in $[T]_\varphi$ for every metric-compatible function φ by Corollary 2.18. We will need a more quantitative version of this fact.

LEMMA 3.6. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation, let $A \subseteq X$ be a measurable subset and let $C > 0$. Then*

$$\mathsf{d}_{\varphi,T}(T_A, \mathrm{id}) \le C\varphi(1)\mu(A) + \sup_{t>C} \frac{\varphi(t)}{t}.$$

Proof. Recall that the T-cocycle of T_A is the return time $n_{T,A}$, which is non-negative. Set $B := \{x \in A\colon n_{T,A}(x) \le C\}$. We have

$$\begin{aligned}\mathsf{d}_{\varphi,T}(T_A, \mathrm{id}) &= \int_B \varphi(n_{T,A}(x)) \, d\mu + \int_{A\setminus B} \varphi(n_{T,A}(x)) \, d\mu \\ &\le C\varphi(1)\mu(B) + \int_{A\setminus B} \frac{\varphi(n_{T,A}(x))}{n_{T,A}} n_{T,A}(x) \, d\mu \\ &\le C\varphi(1)\mu(A) + \left(\sup_{t>C} \frac{\varphi(t)}{t}\right) \int_{A\setminus B} n_{T,A}(x) \, d\mu\end{aligned}$$

and the last integral is at most 1 by Kac's lemma (see equation (2.2)). ∎

COROLLARY 3.7. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Then $\mathsf{d}_{\varphi,T}(T_A, \mathrm{id})$ tends to 0 as $\mu(A)$ approaches 0.*

Proof. Fix $\varepsilon > 0$. By sublinearity, let $C > 0$ be such that for all $t > C$, we have $\varphi(t)/t < \varepsilon$. For all measurable $A \subseteq X$, if $\mu(A) < \varepsilon/C\varphi(1)$, then $\mathsf{d}_{\varphi,T}(T_A, \mathrm{id}) < 2\varepsilon$, which concludes the proof. ∎

REMARK 3.8. In particular, by taking φ bounded, we recover the well-known fact that $d_u(T_A, \mathrm{id})$ tends to 0 as $\mu(A) \to 0$ (see Lemma 4.9).

The following lemma is a direct consequence of Rokhlin's lemma.

LEMMA 3.9. *Let $T \in \mathrm{Aut}(X,\mu)$ be aperiodic and $A \subseteq X$ be measurable. For all $\varepsilon > 0$ and all integers $n \ge 2$, there exists a pre-cycle π of length n, induced by T, such that $\mathrm{supp}(\pi) \subseteq A$ and $\mu(A \setminus \mathrm{supp}(\pi)) \le \varepsilon$.*

Proof. Since T is aperiodic, T_A is aperiodic on its support. We apply Rokhlin's lemma to T_A to find a measurable subset $B \subseteq A$ such that the sets $B, T_A(B), \ldots, (T_A)^{n-1}(B)$ are pairwise disjoint and

$$\mu\big(A \setminus (B \sqcup \cdots \sqcup (T_A)^{n-1}(B))\big) \leq \varepsilon.$$

Then the restriction of T_A to $B \sqcup \cdots \sqcup (T_A)^{n-2}(B)$ is a pre-cycle of length n, which is induced by T_A and thus by T. Finally, its support satisfies the desired assumptions. ∎

We are now ready to prove the existence of n-cycles with prescribed support in φ-integrable full groups.

Proof of Theorem 3.1. Let $A \subseteq X$ be a measurable subset and let $n \geq 2$. Since φ is sublinear, we can and do fix a sequence $(C_k)_{k\geq 1}$ of strictly positive numbers such that

$$\sup_{t>C_k} \frac{\varphi(t)}{t} \leq 2^{-k} \quad \text{for all } k \geq 1.$$

Then, we use Lemma 3.9 to construct inductively a sequence $(\pi_k)_{k\geq 0}$ of pre-cycles of length n induced by T, whose supports are pairwise disjoint subsets of A and such that for all $k \geq 1$,

$$\mu\big(A \setminus (\mathrm{supp}(\pi_0) \sqcup \cdots \sqcup \mathrm{supp}(\pi_{k-1}))\big) \leq \frac{1}{2^k C_k}.$$

This inequality implies in particular that $\mu(\mathrm{supp}(\pi_k)) \leq 1/(2^k C_k)$ for all $k \geq 1$. Let P_k be the closing cycle of π_k and let $P \in \mathrm{Aut}(X,\mu)$ be the n-cycle defined by $P(x) := P_k(x)$ for $x \in \mathrm{supp}(P_k)$ and $P(x) := x$ for $x \notin A$. The support of P is equal to A, and by Lemmas 3.5 and 3.6, we have

$$\begin{aligned}\mathsf{d}_{\varphi,T}(P,\mathrm{id}) &= \sum_{k\geq 0} \mathsf{d}_{\varphi,T}(P_k,\mathrm{id}) \leq 2\sum_{k\geq 0} \mathsf{d}_{\varphi,T}(T_{\mathrm{supp}(\pi_k)},\mathrm{id}) \\ &\leq 2\mathsf{d}_{\varphi,T}(T_{\mathrm{supp}(\pi_0)},\mathrm{id}) + 2\sum_{k\geq 1}\bigg(\varphi(1)C_k\mu(\mathrm{supp}(\pi_k)) + \sup_{t>C_k}\frac{\varphi(t)}{t}\bigg).\end{aligned}$$

The second term is by construction a converging series, so we are done. ∎

3.2. Construction of φ-integrable orbit equivalences. Let us now prove Theorem 1.3.

THEOREM 3.10. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Let $T \in \mathrm{Aut}(X,\mu)$ be ergodic. For all $n \geq 2$, there exists $U \in \mathrm{Aut}(X,\mu)$ such that T and U are φ-integrably orbit equivalent and U^n is not ergodic.*

Proof. By Lemma 2.12, there is a sublinear metric-compatible function ψ such that $\varphi(t) \leq \psi(t)$ for all t large enough. In particular, ψ-integrable orbit equivalence implies φ-orbit equivalence (cf. Remark 2.13). Hence if the theorem holds for ψ then it also holds for φ. Therefore, by replacing φ by ψ, we may and do assume that φ is a metric-compatible function.

By Theorem 3.1, there exists an n-cycle $P \in [T]_\varphi$ whose support is X. We fix a fundamental domain D for P and we let $U := T_D P$. By Lemma 2.3 the following hold:

- the first return maps U_D and T_D coincide: $U_D = T_D$;
- for all $x \in D$, we have $U_D(x) = U^n(x)$;
- T and U have the same orbits.

By the second item, the set D is U^n-invariant. So U^n is not ergodic.

We will now prove that T and U are φ-integrably orbit equivalent. Since T and U have the same orbits, we have to show that $T \in [U]_\varphi$ and $U \in [T]_\varphi$. As a direct consequence of Kac's lemma (see (2.2)), we find that $T_D \in [T]_1 \leq [T]_\varphi$ and therefore $U = T_D P \in [T]_\varphi$.

We now prove that $T \in [U]_\varphi$. In what follows, if a measure-preserving transformation V belongs to $[T] = [U]$, we shall denote by c_V^T the T-cocycle of V and by c_V^U the U-cocycle of V.

Claim. *Let $V \in [T]$. Then for all $y \in D$ such that $V(y) \in D$,*

$$|c_V^U(y)| \leq n |c_V^T(y)|.$$

Proof of Claim. Note that since y and $V(y)$ belong to D, any $i \in \mathbb{Z}$ such that $U^i(z) = V(z)$ must be a multiple of n. If we combine this with the fact that $U_D(z) = U^n(z)$ for all $z \in D$ and $U_D = T_D$, we obtain

$$\begin{aligned}
|c_V^U(y)| &= \min\{|i| \colon U^i(y) = V(y)\} \\
&= n \min\{|i| \colon U_D^i(y) = V(y)\} \\
&= n \min\{|i| \colon T_D^i(y) = V(y)\} \\
&\leq n \min\{|i| \colon T^i(y) = V(y)\} \\
&\leq n |c_V^T(y)|. \quad \blacksquare_{\text{Claim}}
\end{aligned}$$

Let $x \in X$. By definition of U, there are integers $0 \leq k, l \leq n-1$ such that $U^k(x) \in D$ and $U^l(T(x)) \in D$. By the cocycle identity,

$$c^U_{U^l T U^{-k}}(U^k(x)) = c^U_{U^l}(T(x)) + c^U_T(x) + c^U_{U^{-k}}(U^k(x)) = l + c^U_T(x) - k.$$

Hence

$$|c_T^U(x)| \leq |c^U_{U^l T U^{-k}}(U^k(x))| + n.$$

Using the claim for $V = U^l T U^{-k}$ and $y = U^k(x)$, we obtain

$$|c_T^U(x)| \leq n |c^T_{U^l T U^{-k}}(U^k(x))| + n.$$

We now apply φ, use its subadditivity and integrate over X to get

$$\int_X |c_T^U(x)|_\varphi \, d\mu \leq \max_{0 \leq k,l \leq n-1} \int_X n |c^T_{U^l T U^{-k}}(U^k(x))|_\varphi \, d\mu + \varphi(n),$$

which is bounded since $U^lTU^{-k} \in [T]_\varphi$. Hence $T \in [U]_\varphi$ and this concludes the proof of the theorem. ∎

The following direct corollary says that the analogue of Belinskaya's theorem for φ-integrable orbit equivalence does not hold for φ sublinear.

COROLLARY 3.11. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Let $T \in \mathrm{Aut}(X,\mu)$ be an ergodic transformation and assume that T^n is ergodic for some $n \geq 2$. Then there exists $U \in \mathrm{Aut}(X,\mu)$ such that T and U are φ-integrably orbit equivalent but not flip-conjugate.*

Proof. Let $n \geq 2$ be such that T^n is ergodic. By the previous theorem, we find $U \in \mathrm{Aut}(X,\mu)$ such that U is φ-integrably orbit equivalent to T and U^n is not ergodic. In particular, U cannot be flip-conjugate to T because otherwise U^n would be flip-conjugate to T^n, which is ergodic. ∎

REMARK 3.12. Note that T and U do not have the same spectrum, as the spectrum of U contains $\exp(2i\pi/n)$ whereas the spectrum of T does not. So the spectrum is not an invariant of φ-integrable orbit equivalence.

QUESTION 3.13. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $T \in \mathrm{Aut}(X,\mu)$ be an ergodic transformation such that T^n is non-ergodic for all $n \geq 2$. Does there exist $U \in \mathrm{Aut}(X,\mu)$ such that T and U are φ-integrably orbit equivalent but not flip-conjugate?

As we will see in Section 4.5, a weaker result holds in full generality: for every ergodic $T \in \mathrm{Aut}(X,\mu)$ and every sublinear metric-compatible function, there is $U \in [T]_\varphi$ such that U and T have the same orbits, but are not flip-conjugate. This relies on the Baire category theorem, using the fact that $[T]_\varphi$ is a Polish group (see Section 4.1).

3.3. Connection to Shannon orbit equivalence. Let I be a countable set and $f\colon X \to I$ a measurable map. The *Shannon entropy* of f is the quantity

$$H(f) := -\sum_{i\in I} \mu(f^{-1}\{i\}) \log \mu(f^{-1}\{i\}).$$

DEFINITION 3.14 (Kerr–Li). Two aperiodic transformations $T_1, T_2 \in \mathrm{Aut}(X,\mu)$ are *Shannon orbit equivalent* if there exists $S \in \mathrm{Aut}(X,\mu)$ such that ST_1S^{-1} and T_2 have the same orbits and

$$H(c_{ST_1S^{-1}}) < +\infty \quad \text{and} \quad H(c_{T_2}) < +\infty,$$

where $c_{ST_1S^{-1}}$ is the T_2-cocycle of ST_1S^{-1} and c_{T_2} is the ST_1S^{-1}-cocycle of T_2.

LEMMA 3.15. *There are constants $C_1, C_2 > 0$ such that for any measurable function $f\colon X \to \mathbb{Z}$, we have*

$$H(f) \leq C_1 \int_X \log(1 + |f(x)|)\, d\mu + C_2.$$

Proof. The proof we propose is inspired by a classical proof that integrable functions have finite Shannon entropy; see for instance [Aus16a, Lem. 2.1] or [Dow11, Fact 1.1.4].

Let $f_+ := \max(f, 0)$ and $f_- := \min(f, 0)$, so that $f = f_+ + f_-$. We have

$$\int_X \log(1 + |f|)\, d\mu = \int_X \log(1 + f_+)\, d\mu + \int_X \log(1 - f_-)\, d\mu.$$

By subadditivity (see for instance [Dow11, Chap. 1]),

$$H(f) = H(f_+ + f_-) \leq H(f_+) + H(-f_-).$$

Hence, it is enough to prove the lemma for $f\colon X \to \mathbb{N}$. So fix $f : X \to \mathbb{N}$, and for all $n \in \mathbb{N}$, let $p_n := \mu(f^{-1}\{n\})$. By definition of the Shannon entropy,

$$H(f) = -\sum_{n\geq 0} p_n \log p_n.$$

By elementary calculus, one checks that for all $t > 0$ and $s \in \mathbb{R}$, $-t \log t \leq st + e^{-s-1}$. Applying this for $t = p_n$ and $s = 2\log(n + 1)$ and summing over n, we get

$$H(f) \leq 2\sum_{n\geq 0} p_n \log(1 + n) + \sum_{n\geq 0} \frac{e^{-1}}{(n + 1)^2}.$$

To conclude, we observe that $\sum_{n\geq 0} p_n \log(1 + n) = \int_X \log(1 + f(x))\, d\mu$. ■

We immediately deduce the following comparison between φ-integrable orbit equivalence and Shannon orbit equivalence.

THEOREM 3.16. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\log(1 + t) = O(\varphi(t))$ as $t \to +\infty$. Then for any aperiodic transformation $T \in \mathrm{Aut}(X, \mu)$, every $S \in [T]$ whose T-cocycle is φ-integrable has finite Shannon entropy.*

In particular, if two aperiodic transformations $S, T \in \mathrm{Aut}(X, \mu)$ are φ-integrably orbit equivalent, then they are Shannon orbit equivalent.

REMARK 3.17. Note that for every $p \in (0, +\infty)$, we have $\log(1 + t) = O(t^p)$ as $t \to +\infty$. Therefore L^p orbit equivalence implies Shannon orbit equivalence for measure-preserving transformations.

In [KL21], Kerr and Li asked whether Shannon orbit equivalence of ergodic transformations implies flip-conjugacy. We prove that it is not the case.

THEOREM 3.18. *Let $T \in \mathrm{Aut}(X, \mu)$ be an ergodic transformation, assume that T^n is ergodic for some $n \geq 2$. Then there exists $U \in \mathrm{Aut}(X, \mu)$ such that T and U are Shannon orbit equivalent but not flip-conjugate.*

Proof. Consider the sublinear metric-compatible function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ given by $\varphi(t) := \log(1 + t)$. By Corollary 3.11, there exists $U \in \mathrm{Aut}(X, \mu)$ such that T and U are φ-integrably orbit equivalent but not flip-conjugate.

On the other hand, the transformations T and U are Shannon orbit equivalent by Theorem 3.16. ∎

3.4. Finiteness of entropy and Shannon orbit equivalence. Kerr and Li implicitly asked whether dynamical entropy is an invariant of Shannon orbit equivalence for ergodic measure-preserving transformations. Shortly after a first version of our paper appeared, they obtained a positive answer [KL22, Thm. A]. In this section, we provide a short proof that *finiteness* of dynamical entropy is an invariant of Shannon orbit equivalence. We start by recalling a definition of dynamical entropy of measure-preserving transformations which is convenient for our purposes.

DEFINITION 3.19. Let $T \in \mathrm{Aut}(X,\mu)$. A measurable map $f\colon X \to I$, where I is countable, is called *T-dynamically generating* if there is a full measure set $X_0 \subseteq X$ such that for all distinct $x, y \in X_0$, there is $n \in \mathbb{Z}$ such that $f(T^n(x)) \neq f(T^n(y))$.

DEFINITION 3.20. The *dynamical entropy* of a measure-preserving transformation $T \in \mathrm{Aut}(X,\mu)$ is the infimum of the Shannon entropies of its T-dynamically generating functions.

The above definition is not the standard definition, but it is equivalent to it by a theorem of Rokhlin [Rok67, Thm. 10.8]. Also note that by definition, the dynamical entropy of $T \in \mathrm{Aut}(X,\mu)$ is finite if and only if T admits a dynamically generating function of finite entropy.

PROPOSITION 3.21. *Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation with infinite dynamical entropy and let $U \in [T]$ be a transformation whose T-cocycle has finite Shannon entropy. Then U has infinite dynamical entropy.*

Proof. Let $f\colon X \to I$ be a U-dynamically generating function and denote by c_U the T-cocycle of U. We claim that the couple $(f, c_U)\colon X \to I \times \mathbb{Z}$ is T-dynamically generating. Indeed, let $x, y \in X$ be such that

$$c_U(T^n(x)) = c_U(T^n(y)) \quad \text{and} \quad f(T^n(x)) = f(T^n(y)) \quad \text{for all } n \in \mathbb{Z}.$$

The first equality and the cocycle identity imply that $c_{U^n}(x) = c_{U^n}(y)$ for all $n \in \mathbb{Z}$. So for all $n \in \mathbb{Z}$,

$$f(U^n(x)) = f(T^{c_{U^n}(x)}(x)) = f(T^{c_{U^n}(y)}(y)) = f(U^n(y)).$$

Since f is U-dynamically generating, the above equation implies that $x = y$.

Assume for contradiction that U has finite entropy. This implies that there exists a U-generating function f with finite Shannon entropy. Since c_U has finite Shannon entropy, so does (f, c_U). But we have seen that (f, c_U) is a T-generating function, hence T has finite dynamical entropy and the proof is complete. ∎

COROLLARY 3.22. *Suppose T_1 and T_2 are aperiodic measure-preserving transformations which are Shannon orbit equivalent. Then T_1 has finite dynamical entropy if and only if T_2 has finite dynamical entropy.*

REMARK 3.23. As explained before, Kerr and Li recently proved that dynamical entropy itself is preserved under Shannon orbit equivalence [KL22, Thm. A]. Suppose now that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\log(1+t) = O(\varphi(t))$ as $t \to +\infty$, for example $\varphi(t) = \log(1+t)$ or $\varphi(t) = t^p$. By Kerr and Li's result and Theorem 3.16, dynamical entropy is an invariant of φ-integrable orbit equivalence. In particular, it is an invariant of L^p orbit equivalence for $p \in (0, +\infty)$. When φ is moreover sublinear, this is the only invariant of φ-integrable orbit equivalence that we know for ergodic transformations, even for L^p orbit equivalence with $p \in (0, 1)$.

4. Weakly mixing elements are generic in $[T]_\varphi$. This last section is dedicated to the proof of Theorem 1.2: we are going to show that for every sublinear metric-compatible function φ and ergodic transformation T, there is an element $U \in [T]_\varphi$ which has the same orbit as T but is not flip-conjugate to T.

Note that we have shown in Corollary 3.11 that this is already the case if T is an ergodic transformation such that T^n is ergodic for some $n \geq 2$. Therefore we will restrict ourselves to the case when there exists $n \geq 2$ such that T^n is not ergodic. For such transformations, we will not construct any explicit $U \in [T]_\varphi$, but we will use the Baire category theorem: we will show that given an aperiodic transformation T, the possible candidates for such U are generic; see Theorem 4.15.

We start with three preparatory sections to introduce the required material. We believe them to be of independent interest. In the first one, we show that $\mathsf{d}_{\varphi,T}$ is a complete separable metric inducing a Polish group topology on $[T]_\varphi$; see Theorem 4.1. In the second one, we prove a sublinear ergodic theorem in the context of φ-integrability, which will play a crucial role later on. In the third one, we study continuity properties of the first return map.

4.1. Polish group topology. Recall that the full group $[T] \leq \mathrm{Aut}(X, \mu)$ is closed and separable for the topology induced by the uniform metric d_u and therefore it is a Polish group [Kec10, Prop. 3.2]. We shall see that φ-integrable full groups provide further interesting classes of Polish groups.

Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function and let $T \in \mathrm{Aut}(X, \mu)$ be an aperiodic transformation. We introduced the φ-integrable full group $[T]_\varphi$ as the group of measure-preserving transformation whose cocycle is φ-integrable. In Lemma 2.14, we defined a metric $\mathsf{d}_{\varphi,T}$ on $[T]_\varphi$. The goal of this section is to prove that the topology induced by $\mathsf{d}_{\varphi,T}$ on $[T]_\varphi$ is a Polish group topology.

THEOREM 4.1. *The metric $\mathsf{d}_{\varphi,T}$ is complete, separable and right-invariant on $[T]_\varphi$ and the topology generated by $\mathsf{d}_{\varphi,T}$ is a group topology. In particular, $[T]_\varphi$ is a Polish group.*

Proof. We have already shown in Lemma 2.14 that $\mathsf{d}_{\varphi,T}$ is right-invariant. Corollary 2.18 tells us that the inclusion $[T]_\varphi \hookrightarrow [T]$ is Lipschitz and in particular any $\mathsf{d}_{\varphi,T}$-Cauchy sequence is d_u-Cauchy. Since d_u is complete, any $\mathsf{d}_{\varphi,T}$-Cauchy sequence has a d_u-limit.

CLAIM. *Let $(U_n)_{n\geq 0}$ be a $\mathsf{d}_{\varphi,T}$-Cauchy sequence of elements of $[T]_\varphi$ and let $U \in [T]$ be its d_u-limit. Then $U \in [T]_\varphi$ and $\lim_n \mathsf{d}_{\varphi,T}(U_n, U) = 0$. In particular, $\mathsf{d}_{\varphi,T}$ is complete.*

Proof of Claim. Since $(U_n)_{n\geq 0}$ is $\mathsf{d}_{\varphi,T}$-Cauchy, there is m such that for all $n \geq m$,

$$\begin{aligned}\int_X |c_{U_n}(x)|_\varphi \, d\mu &= \mathsf{d}_{\varphi,T}(U_n, \mathrm{id})\\ &\leq \mathsf{d}_{\varphi,T}(U_n, U_m) + \mathsf{d}_{\varphi,T}(U_m, \mathrm{id}) \leq 1 + \mathsf{d}_{\varphi,T}(U_m, \mathrm{id}).\end{aligned}$$

Moreover, since $\lim_n d_u(U_n, U) = 0$, $(c_{U_n})_{n\geq 0}$ converges in measure to c_U and thus a subsequence converges pointwise to c_U. Fatou's lemma then implies that $U \in [T]_\varphi$. The triangle inequality for $|\cdot|_\varphi$ gives

$$\begin{aligned}\int_X \big||c_{U_n}(x) - c_U(x)|_\varphi - |c_{U_m}(x) - c_U(x)|_\varphi\big| \, d\mu &\leq \int_X |c_{U_n}(x) - c_{U_m}(x)|_\varphi \, d\mu\\ &= \mathsf{d}_{\varphi,T}(U_n, U_m),\end{aligned}$$

hence $(|c_{U_n} - c_U|_\varphi)_{n\geq 0}$ is a Cauchy sequence with respect to the L^1 metric. Since $(c_{U_n} - c_U)_{n\geq 0}$ converges in measure to 0, we must have

$$\lim_n \mathsf{d}_{\varphi,T}(U_n, U) = \lim_n \int_X |c_{U_n}(x) - c_U(x)|_\varphi \, d\mu = 0$$

so the claim is proved. $\blacksquare_{\text{Claim}}$

Let us now show that the topology induced by $\mathsf{d}_{\varphi,T}$ is a group topology. We start by proving the continuity of the inverse map. Let $(U_n)_{n\geq 0}$ be a sequence of elements of $[T]_\varphi$ converging to $U \in [T]_\varphi$. Then the cocycle identity gives $0 = c_{UU^{-1}}(x) = c_U(U^{-1}(x)) + c_{U^{-1}}(x)$ and hence

$$\begin{aligned}\mathsf{d}_{\varphi,T}(U_n^{-1}, U^{-1}) &= \int_X |c_{U_n^{-1}}(x) - c_{U^{-1}}(x)|_\varphi \, d\mu\\ &= \int_X |c_{U_n}(U_n^{-1}(x)) - c_U(U^{-1}(x))|_\varphi \, d\mu\end{aligned}$$

$$
\begin{aligned}
&= \int_X |c_{U_n}(x) - c_U(U^{-1}U_n(x))|_\varphi \, d\mu \\
&\le \int_X |c_{U_n}(x) - c_U(x)|_\varphi \, d\mu + \int_X |c_U(x) - c_U(U^{-1}U_n(x))|_\varphi \, d\mu.
\end{aligned}
$$

Since $(U_n)_{n\geq 0}$ converges to U for the metric $\mathsf{d}_{\varphi,T}$ and thus for the uniform metric, the right hand side converges to 0 and hence the inverse map is continuous.

We now prove that the multiplication map is continuous. Let $(U_n)_{n\geq 0}$ and $(V_m)_{m\geq 0}$ be sequences which $\mathsf{d}_{\varphi,T}$-converge to U and V respectively. Then by the triangle inequality and right-invariance,

$$
\mathsf{d}_{\varphi,T}(U_nV_n, UV) \le \mathsf{d}_{\varphi,T}(U_n, U) + \mathsf{d}_{\varphi,T}(UV_n, UV).
$$

Now observe that since the inverse map is continuous, UV_n converges to UV if and only if $V_n^{-1}U^{-1}$ converges to $V^{-1}U^{-1}$. By right-invariance and continuity of the inverse, $\lim_n \mathsf{d}_{\varphi,T}(V_n^{-1}U^{-1}, V^{-1}U^{-1}) = 0$, which finishes the proof that $\mathsf{d}_{\varphi,T}$ induces a group topology on $[T]_\varphi$.

It remains to show that this topology is separable. Consider the following abelian group where we identify functions up to a null set:

$$
\mathrm{L}^\varphi(X,\mathbb{Z}) := \Big\{ f\colon X \to \mathbb{Z}\colon \int_X |f(x)|_\varphi \, d\mu < +\infty \Big\},
$$

endowed with the metric $(f,g) \mapsto \int_X |f(x) - g(x)|_\varphi \, d\mu$. The function which takes $U \in [T]_\varphi$ to $c_U \in \mathrm{L}^\varphi(X,\mathbb{Z})$ is an isometry. So $[T]_\varphi$ is isometric to a metric subspace of $\mathrm{L}^\varphi(X,\mathbb{Z})$. We now prove that $\mathrm{L}^\varphi(X,\mathbb{Z})$ is separable: identify X with $[0,1]$ equipped with the Lebesgue measure and observe that the subgroup generated by the characteristic functions of rational intervals is dense. Since subspaces of separable metric spaces are separable, we conclude that $[T]_\varphi$ is separable. ∎

We now exploit the Polish group topology to characterize the inclusion between φ-integrable full groups in terms of metric comparisons. In particular, $[T]_\varphi \neq [T]_\psi$ as long as φ and ψ are not bi-Lipschitz. However, we do not know how to construct any explicit element in $[T]_\varphi \setminus [T]_\psi$.

Proposition 4.2. *Let φ and ψ be metric-compatible functions and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic measure-preserving transformation. Then the following are equivalent:*

(i) *$\varphi(t) = O(\psi(t))$ as $t \to +\infty$.*
(ii) *$[T]_\psi \le [T]_\varphi$.*

The proof uses the following well-known lemma.

Lemma 4.3. *Let G be a Polish group, and let $H_1 \le H_2$ be subgroups of G. Suppose that H_1 and H_2 are endowed with a Polish topology which refines the* *topology induced by G. Then the topology of H_1 refines the topology induced by H_2.*

Proof. By hypothesis, the inclusions $H_1 \hookrightarrow G$ and $H_2 \hookrightarrow G$ are continuous. In particular, the Borel structure induced by each of their topologies refines the Borel structure induced by the one of G. The Luzin–Suslin theorem states that given any Polish spaces X and Y, if $f\colon X \to Y$ is Borel and injective then for every Borel $A \subseteq X$, the set $f(A)$ is Borel (see [Kec95, Thm. 15.1]). Therefore, we can apply it to the inclusions $H_1 \hookrightarrow G$ and $H_2 \hookrightarrow G$ to deduce that the Borel structures induced by the respective topologies of H_1 and H_2 coincide with the σ-algebra induced by the Borel subsets of G. This in particular tells us that the inclusion map $H_1 \hookrightarrow H_2$ is Borel, so it is automatically continuous by Pettis' lemma [Kec95, Thm. 9.9], which completes the proof. ■

Proof of Proposition 4.2. The implication (i)⇒(ii) follows from Lemma 2.17, so we only need to prove (ii)⇒(i). Towards a contradiction, assume that $[T]_\psi \leq [T]_\varphi$ but (i) does not hold. By Lemma 2.16, there exists a sequence $(k_n)_{n\geq 0}$ of positive integers such that

$$\lim_{n\to+\infty} \frac{\psi(k_n)}{\varphi(k_n)} = 0. \tag{4.1}$$

Corollary 2.18 tells us that $[T]_\varphi$ and $[T]_\psi$ embed continuously in $[T]$. Therefore Lemma 4.3 implies that the inclusion map of $[T]_\psi$ into $[T]_\varphi$ is continuous. We will obtain a contradiction by constructing a sequence $(U_n)_{n\geq 0}$ of elements of $[T]_\psi$ such that

$$\mathsf{d}_{\psi,T}(U_n, \mathrm{id}) \to 0 \quad \text{but} \quad \mathsf{d}_{\varphi,T}(U_n, \mathrm{id}) \not\to 0.$$

By Rokhlin's lemma, for every $n \in \mathbb{N}$ one can find a measurable subset $A_n \subseteq X$ such that $A_n, T(A_n), \ldots, T^{2k_n-1}(A_n)$ are pairwise disjoint and $\mu(A_n) \geq \frac{1}{4k_n}$. Note that

$$\mu\Big(\bigsqcup_{i=0}^{k_n-1} T^i(A_n)\Big) \geq \frac{1}{4}.$$

Hence, for all n such that $\varphi(k_n) \geq 4$, we can pick a measurable subset $B_n \subseteq \bigsqcup_{i=0}^{k_n-1} T^i(A_n)$ of measure exactly $\frac{1}{\varphi(k_n)}$. We then define $U_n \in [T]_\varphi$ by

$$U_n(x) := \begin{cases} T^{k_n}(x) & \text{if } x \in B_n, \\ T^{-k_n}(x) & \text{if } x \in T^{k_n}(B_n), \\ x & \text{otherwise.} \end{cases}$$

By construction $\mathsf{d}_{\varphi,T}(U_n, \mathrm{id}) = 2\mu(B_n)\varphi(k_n) = \frac{1}{2}$, but equation (4.1) implies that $d_{\psi,T}(U_n, \mathrm{id}) = 2\mu(B_n)\psi(k_n) \to 0$, a contradiction. ■

COROLLARY 4.4. *Let φ and ψ be metric-compatible functions, and let $T \in \mathrm{Aut}(X,\mu)$ be aperiodic. Then $[T]_\varphi = [T]_\psi$ if and only if $\varphi(t) = O(\psi(t))$ and $\psi(t) = O(\varphi(t))$ as $t \to +\infty$.*

4.2. A sublinear ergodic theorem for φ-integrable functions. In this section, we prove the following sublinear ergodic theorem, which will be a key tool in our analysis of the first return map. Given a measurable function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, a measurable function $f : X \to \mathbb{C}$ is *φ-integrable* when $\int_X \varphi(|f(x)|)\, d\mu < +\infty$.

THEOREM 4.5. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $U \in \mathrm{Aut}(X,\mu)$ and let $f\colon X \to \mathbb{C}$ be a measurable function which is φ-integrable. Then for almost every $x \in X$,*

$$\lim_n \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|\Big) = 0.$$

The convergence also holds in L^1, that is,

$$\lim_n \int_X \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|\Big)\, d\mu = 0.$$

Proof. Given $n \geq 1$, let

$$g_n(x) := \varphi\Big(\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|\Big) \quad \text{for all } x \in X.$$

Using the fact that φ is metric-compatible, we deduce that the sequence $(g_n)_{n\geq 1}$ of functions has Kingman's subadditivity property: for all $n, m \geq 1$ and all $x \in X$,

$$g_{n+m}(x) \leq g_n(x) + g_m(U^n(x)).$$

Kingman's subadditive theorem [Kin68] implies that $(g_n/n)_{n\geq 0}$ converges almost everywhere to some function h and our aim becomes to show that $h = 0$. Recall that a sequence that converges in L^1 admits an almost surely converging subsequence. In order to prove that $h = 0$, it is therefore enough to prove that $\|g_n/n\|_1$ converges to 0, that is, to establish the second part of the theorem.

To this end, let $\varepsilon > 0$. Since $\varphi(|f|)$ is integrable and φ is non-decreasing, we find a measurable subset $A \subseteq X$ and $K \geq 0$ such that $\int_{X\setminus A} \varphi(|f(x)|)\, d\mu \leq \varepsilon$ and $|f(x)| \leq K$ for every $x \in A$. For every measurable subset $B \subseteq X$, we denote $f_B := f\mathbb{1}_B$, where $\mathbb{1}_B$ is the indicator function of B. With this notation, using first the fact that φ is subadditive non-decreasing and then that U preserves the measure, we obtain

$$\begin{aligned}\limsup_n \int_X \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f_{X\setminus A}(U^k(x))\Big|\Big)\,d\mu \\ &\le \limsup_n \int_X \frac{1}{n}\sum_{k=0}^{n-1}\varphi\big(|f_{X\setminus A}(U^k(x))|\big)\,d\mu \\ &= \int_X \varphi(|f_{X\setminus A}(x)|)\,d\mu \le \varepsilon.\end{aligned}$$

Moreover, since f_A is bounded by K, for all $x\in X$ we have

$$\frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f_A(U^k(x))\Big|\Big) \le \frac{\varphi(nK)}{n} = K\frac{\varphi(nK)}{nK}.$$

Integrating over X, we obtain

$$\int_X \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f_A(U^k(x))\Big|\Big)\,d\mu \le K\frac{\varphi(nK)}{nK}.$$

Using $f = f_{X\setminus A} + f_A$ and subadditivity, we deduce

$$\limsup_n \int_X \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|\Big)\,d\mu \le \varepsilon + \limsup_n K\frac{\varphi(nK)}{nK}.$$

Since φ is sublinear, we finally obtain

$$\limsup_n \int_X \frac{1}{n}\varphi\Big(\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|\Big)\,d\mu \le \varepsilon.$$

This proves that $\|g_n/n\|_1 \to 0$, thus ending the proof of the theorem. ∎

Here is our main application, which will be a key tool in the following section.

Corollary 4.6. *Let φ be a sublinear metric-compatible function and let $T\in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Then for every $U\in [T]_\varphi$,*

$$\lim_n \frac{\mathsf{d}_{\varphi,T}(U^n,\mathrm{id})}{n} = 0.$$

Proof. For all integers $n\ge 0$ and all $x\in X$, by the cocycle identity and the triangle inequality we have

$$|c_{U^n}(x)| \le \sum_{k=0}^{n-1} |c_U(U^k(x))|.$$

We apply Theorem 4.5 to the function $f(x) := |c_U(x)|$ to get

$$\frac{\mathsf{d}_{\varphi,T}(U^n,\mathrm{id})}{n} \leq \int_X \frac{1}{n}\Big|\sum_{k=0}^{n-1} f(U^k(x))\Big|_\varphi \xrightarrow[n\to\infty]{} 0. \ \blacksquare$$

REMARK 4.7. We do not fully understand the asymptotics of the sequence $(\mathsf{d}_{\varphi,T}(U^n,\mathrm{id}))_{n\geq 0}$. For instance, when does the sequence $(\mathsf{d}_{\varphi,T}(U^n,\mathrm{id})/\varphi(n))_{n\geq 0}$ converge?

4.3. Continuity properties of the first return map. In this section we are primarily interested in continuity properties of the first return map. An important preliminary step is the following analogue of Kac's lemma, saying that φ-integrable full groups are stable under first return maps.

LEMMA 4.8. *Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation and let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. For all $U \in [T]_\varphi$ and all measurable subsets $A \subseteq X$, we have $\mathsf{d}_{\varphi,T}(U_A,\mathrm{id}) \leq \mathsf{d}_{\varphi,T}(U,\mathrm{id})$. In particular, $U_A \in [T]_\varphi$.*

Proof. For every integer $j \geq 1$, set $A_j := \{x \in A\colon n_{U,A}(x) = j\}$ where $n_{U,A}$ is the first return time of U to A as defined in Section 2. Then

$$\int_X |c_{U_A}(x)|_\varphi\, d\mu = \int_A |c_{U_A}(x)|_\varphi\, d\mu = \sum_{j=1}^{+\infty} \int_{A_j} |c_{U_A}(x)|_\varphi\, d\mu = \sum_{j=1}^{+\infty} \int_{A_j} |c_{U^j}(x)|_\varphi\, d\mu.$$

By the cocycle identity, for every $j \geq 1$ we have $c_{U^j}(x) = \sum_{i=0}^{j-1} c_U(U^i(x))$, so by the triangle inequality we obtain

$$\begin{aligned}\int_X |c_{U_A}(x)|_\varphi\, d\mu &\leq \sum_{j=1}^{+\infty}\sum_{i=0}^{j-1} \int_{A_j} |c_U(U^i(x))|_\varphi\, d\mu \\ &\leq \sum_{j=1}^{+\infty}\sum_{i=0}^{j-1} \int_{U^i(A_j)} |c_U(x)|_\varphi\, d\mu \leq \int_X |c_U(x)|_\varphi\, d\mu,\end{aligned}$$

the last inequality being a consequence of the fact that the sets $U^i(A_j)$ are pairwise disjoint for $j \in \mathbb{N}$ and $i \in \{0,\dots,j-1\}$. $\blacksquare$

In order to state the continuity properties of the first return map, let us first observe that since we are working up to measure zero, the first return map with respect to a set A only depends on A up to a null set. It is therefore natural to introduce the measure algebra $\mathrm{MAlg}(X,\mu)$, defined as the algebra of measurable subsets modulo identifying subsets which differ by a null set. We endow $\mathrm{MAlg}(X,\mu)$ with the metric $d_\mu(A,B) := \mu(A \triangle B)$.

We can now recall a continuity property satisfied by the first return map in the full group, which was first observed by Keane.

LEMMA 4.9 ([Kea70, Lem. 3]). *Let T be a measure-preserving transformation. Then the map*

$$[T] \times \mathrm{MAlg}(X,\mu) \to [T], \qquad (U,A) \mapsto U_A,$$

is continuous.

It is worth noting that the analogue of Lemma 4.8 fails for the L^1-full group. Indeed, let $T \in \mathrm{Aut}(X,\mu)$ be ergodic and let $\varphi := \mathrm{id}_{\mathbb{R}_+}$ Then Kac's lemma implies that for all measurable $A \subseteq X$ of positive measure, $\mathsf{d}_{\varphi,T}(T_A,\mathrm{id}) = d_{\varphi,T}(T,\mathrm{id}) = 1$. Since $T_\emptyset = \mathrm{id}$, this shows that the map $\mathrm{MAlg}(X,\mu) \to [T]_1$ defined by $A \mapsto T_A$ is not continuous.

However, the situation is not that clear when φ is sublinear.

QUESTION 4.10. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a *sublinear* metric-compatible function. Is the map $\mathrm{MAlg}(X,\mu) \to [T]_\varphi$ defined by $A \mapsto T_A$ continuous? More generally, is the map $[T]_\varphi \times \mathrm{MAlg}(X,\mu) \to [T]_\varphi$ given by $(U,A) \mapsto U_A$ continuous?

In this section, we give two partial answers to the above questions. We first prove that the map $A \mapsto U_A$ satisfies a continuity property "from below". For this, we need the following version of Scheffé's lemma for sequences of $\mathbb{Z}$-valued φ-integrable functions.

LEMMA 4.11. *Let $f\colon X \to \mathbb{Z}$ be a measurable function and let $(f_n)_{n\geq 0}$ be a sequence of measurable functions $f_n\colon X \to \mathbb{Z}$ that converge in measure to f. If*

$$\limsup \int_X |f_n|_\varphi \, d\mu \leq \int_X |f|_\varphi \, d\mu,$$

then $\lim_n \int_X |f_n - f|_\varphi \, d\mu = 0$.

Proof. It suffices to show that given $\varepsilon > 0$, there is $\delta > 0$ such that for all measurable functions $g\colon X \to \mathbb{Z}$ satisfying

$$\mu(\{x \in X : f(x) \neq g(x)\}) \leq \delta \quad \text{and} \quad \int_X |g|_\varphi \, d\mu \leq \int_X |f|_\varphi \, d\mu + \delta, \tag{4.2}$$

one has $\int_X |f-g|_\varphi \, d\mu \leq \varepsilon$. To this end, fix $\varepsilon > 0$. Since $\int_X |f|_\varphi \, d\mu < +\infty$, by Lebesgue's dominated convergence theorem there exists $\delta_0 > 0$ such that for all measurable subsets $A \subseteq X$, if $\mu(A) < \delta_0$ then $\int_A |f|_\varphi \, d\mu < \varepsilon$. Take $\delta := \min\{\delta_0, \varepsilon\}$. Let $g\colon X \to \mathbb{Z}$ be a measurable function satisfying (4.2). If we let $A := \{x \in X\colon f(x) \neq g(x)\}$, we have

$$\int_A |g|_\varphi \, d\mu = \int_X |g|_\varphi \, d\mu - \int_{X\setminus A} |g|_\varphi \, d\mu \leq \int_X |f|_\varphi \, d\mu - \int_{X\setminus A} |f|_\varphi \, d\mu + \delta \leq 2\varepsilon,$$

and we can therefore conclude the proof:

$$\int_X |f-g|_\varphi \, d\mu = \int_A |f-g|_\varphi \, d\mu \leq \int_A |f|_\varphi \, d\mu + \int_A |g|_\varphi \, d\mu \leq 3\varepsilon. \blacksquare$$

We can now prove the following proposition, which is the φ-integrable analogue of [LM18, Prop. 3.9].

PROPOSITION 4.12. *Let φ be a metric-compatible function and $T \in \mathrm{Aut}(X,\mu)$ an ergodic transformation. Let $U \in [T]_\varphi$ and consider a measurable subset $A \subseteq X$. If $(A_n)_{n\geq 0}$ is a sequence of measurable subsets of A such that* $\lim_n \mu(A \setminus A_n) = 0$, *then* $\lim_n \mathsf{d}_{\varphi,T}(U_{A_n}, U_A) = 0$.

Proof. Since $\lim_n \mu(A \setminus A_n) = 0$ and since the first return map is continuous with respect to the uniform metric by Lemma 4.9, we get $\lim_n d_u(U_{A_n}, U_A) = 0$. This means that $(c_{U_{A_n}})_{n\geq 0}$ converges in measure to c_{U_A} and therefore $(|c_{U_{A_n}}|_\varphi)_{n\geq 0}$ converges in measure to $|c_{U_A}|_\varphi$. Thanks to Lemma 4.8, we have $d_{\varphi,T}(U_{A_n}, \mathrm{id}) \leq d_{\varphi,T}(U_A, \mathrm{id})$ for all $n \geq 0$. In other words,

$$\int_X |c_{U_{A_n}}(x)|_\varphi \, d\mu \leq \int_X |c_{U_A}(x)|_\varphi \, d\mu.$$

Hence we can apply Lemma 4.11, yielding

$$\lim_n \int_X |c_{U_{A_n}}(x) - c_{U_A}(x)|_\varphi \, d\mu = 0.$$

This precisely means that $\lim_n \mathsf{d}_{\varphi,T}(U_{A_n}, U_A) = 0$. ■

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function and $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. In Corollary 3.7, we proved that for any aperiodic transformation $T \in \mathrm{Aut}(X,\mu)$, the quantity $\mathsf{d}_{\varphi,T}(T_A, \mathrm{id})$ tends to 0 as $\mu(A) \to 0$. It is natural to ask whether this holds for all aperiodic $U \in [T]_\varphi$, i.e. does $\mathsf{d}_{\varphi,T}(U_A, \mathrm{id})$ tend to 0 as $\mu(A) \to 0$? We have not been able to answer this question, but we can prove the following much weaker statement. Its proof relies on our sublinear ergodic theorem (Theorem 4.5), or rather on Corollary 4.6.

PROPOSITION 4.13. *Let φ be a sublinear metric-compatible function. Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Then for any aperiodic transformation $U \in [T]_\varphi$ and for any measurable subset $A \subseteq X$, there exists a sequence $(A_k)_{k\geq 0}$ of measurable subsets contained in A which intersect every U-orbit and such that* $\lim_k \mu(A_k) = 0$ *and* $\lim_k \mathsf{d}_{\varphi,T}(U_{A_k}, \mathrm{id}) = 0$.

Proof. Put $V := U_A$ and note that for every measurable $B \subseteq A$, we have $V_B = U_B$. As an immediate consequence of Alpern's multiple Rokhlin theorem [Alp79] ([2]), for every $k \geq 0$ one can find a measurable subset $B_k \subseteq A$ which meets every V-orbit in A and such that $n_{V,B_k}(B_k) = \{k, k+1\}$. The latter implies that the $V^i(B_k)$ are pairwise disjoint for $i \in \{0, \dots, k-1\}$. Observe that for all $x \in X$ and $i \in \mathbb{Z}$ we have $n_{V,V^i(B_k)}(x) = n_{V,B_k}(V^{-i}(x))$.

([2]) We actually only need Step 1 from the simpler proof given in [EP97].

This implies that for all $x \in V^i(B_k)$, either $V_{V^i(B_k)}(x) = V^k(x)$ or $V_{V^i(B_k)}(x) = V^{k+1}(x)$. Therefore by integrating over the disjoint union of the $V^i(B_k)$ for $i \in \{0, \ldots, k-1\}$ we get

$$\sum_{i=0}^{k-1} \mathsf{d}_{\varphi,T}(V_{V^i(B_k)}, \mathrm{id}) \leq \mathsf{d}_{\varphi,T}(V^k, \mathrm{id}) + \mathsf{d}_{\varphi,T}(V^{k+1}, \mathrm{id}),$$

whence there exists $0 \leq i_k \leq k-1$ such that

$$\mathsf{d}_{\varphi,T}(V_{V^{i_k}(B_k)}, \mathrm{id}) \leq \frac{\mathsf{d}_{\varphi,T}(V^k, \mathrm{id}) + \mathsf{d}_{\varphi,T}(V^{k+1}, \mathrm{id})}{k}.$$

The set $A_k := V^{i_k}(B_k)$ has measure less than $1/k$. Corollary 4.6 implies that the right hand side in the above formula tends to zero, which implies

$$\lim_k \mathsf{d}_{\varphi,T}(U_{A_k}, \mathrm{id}) = \lim_k \mathsf{d}_{\varphi,T}(V_{A_k}, \mathrm{id}) = 0. \ \blacksquare$$

4.4. Optimality of Belinskaya's theorem. We are now ready to prove Theorem 1.2: for any sublinear function φ, Belinskaya's theorem fails if we replace integrability by φ-integrability.

THEOREM 4.14. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function and let $T_1 \in \mathrm{Aut}(X, \mu)$ be ergodic. Then there exists an ergodic transformation $T_2 \in [T_1]$ whose cocycle is φ-integrable such that T_1 and T_2 have the same orbits but are not flip-conjugate.*

The proof of the theorem depends on whether T_1 is weakly mixing or not. Indeed, if it is, then we can use Corollary 3.11. Otherwise, we have to use the Baire category theorem. Indeed, the candidate for T_2 is generic for the topology induced by d_{φ,T_1}.

THEOREM 4.15. *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X, \mu)$ be an aperiodic element. Then the set of all elements of $[T]_\varphi$ which are weakly mixing and have the same orbits as T is a dense G_δ set in the Polish space of aperiodic elements of $[T]_\varphi$ with the topology induced by $\mathsf{d}_{\varphi,T}$.*

We delay the proof of the above theorem to Section 4.5. Let us first explain how to deduce Theorem 4.14 from Theorem 4.15.

Proof of Theorem 4.14. By Lemma 2.12, there is a sublinear metric-compatible function ψ such that $\varphi(t) \leq \psi(t)$ for all t large enough. In particular, ψ-integrability implies φ-integrability for $\mathbb{Z}$-valued functions (cf. Remark 2.13). Hence if the theorem holds for ψ then it holds for φ. Therefore, by replacing φ by ψ, we may and do assume that φ is metric-compatible.

If T_1 is weakly mixing, then all its nontrivial powers are ergodic. Thus Corollary 3.11 implies that there exists $T_2 \in [T_1]_\varphi$ such that T_1 and T_2 have the same orbits but are not flip-conjugate.

If T_1 is not weakly mixing, then Theorem 4.15 grants us some weakly mixing $T_2 \in [T_1]_\varphi$ such that T_1 and T_2 have the same orbits. Since T_2 is weakly mixing and T_1 is not, they cannot be flip-conjugate. ∎

4.5. The weakly mixing elements form a dense G_δ set. This section is dedicated to the proof of Theorem 4.15. Before starting the proof, we will need some terminology and preliminary propositions.

In this section, we will consider the φ-integrable full groups, both with the topology induced by the uniform metric d_u and with their natural topology induced by $\mathsf{d}_{\varphi,T}$. The metric $\mathsf{d}_{\varphi,T}$ is complete so we can apply the Baire category theorem in $([T]_\varphi, \mathsf{d}_{\varphi,T})$ (see Theorem 4.1). Moreover, the topology induced by $\mathsf{d}_{\varphi,T}$ refines the topology induced by d_u (see Corollary 2.18). Note that $([T]_\varphi, d_u)$ is not complete; indeed, one can show that $[T]_\varphi$ is dense in the complete metric space $([T], d_u)$.

Denote by $\mathrm{APER} \subseteq \mathrm{Aut}(X,\mu)$ the set of aperiodic transformations.

Lemma 4.16. *Let φ be a metric-compatible function and let $T \in \mathrm{APER}$. Then the set $\mathrm{APER} \cap [T]_\varphi$ is closed in the complete metric space $([T]_\varphi, \mathsf{d}_{\varphi,T})$ and hence it is a complete metric space itself.*

Proof. Note that T is aperiodic if and only if for all $n \geq 1$ we have $d_u(T^n, \mathrm{id}) = 1$. So the set APER is closed in $(\mathrm{Aut}(X,\mu), d_u)$. In particular, $\mathrm{APER} \cap [T]_\varphi$ is closed in $([T]_\varphi, d_u)$, so it is also closed in $([T]_\varphi, \mathsf{d}_{\varphi,T})$. ∎

Proposition 4.17. *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{APER}$. Then the set*

$$\{U \in \mathrm{APER} \cap [T]_\varphi \colon T \text{ and } U \text{ have the same orbits}\}$$

is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \mathsf{d}_{\varphi,T})$.

Proof. We first prove that this set is G_δ. For all $\varepsilon > 0$ and $n \geq 1$, let

$$O_{\varepsilon,n} := \big\{U \in [T]_\varphi \colon \mu(\{x \in X \colon T(x) \in \{U^{-n}(x), \dots, U^n(x)\}\}) > 1 - \varepsilon\big\}.$$

Each $O_{\varepsilon,n}$ is open in $([T]_\varphi, d_u)$ and thus also in $([T]_\varphi, \mathsf{d}_{\varphi,T})$. Moreover,

$$\{U \in \mathrm{APER} \cap [T]_\varphi \colon T \text{ and } U \text{ have the same orbits}\} = \bigcap_{\varepsilon \in \mathbb{Q}_+^*} \bigcup_{n \geq 1} O_{\varepsilon,n},$$

which is a countable intersection of open sets, and thus a G_δ set.

We now prove the density. Let $U \in \mathrm{APER} \cap [T]_\varphi$. Fix a sequence $(A_k)_{k\geq 0}$ of measurable subsets of X which intersect every S-orbit, such that $\lim_k \mu(A_k) = 0$ and $\lim_k \mathsf{d}_{\varphi,T}(U_{A_k}, \mathrm{id}) = 0$ as in Proposition 4.13. If we set $P_k := (U_{A_k})^{-1} U$, then $(P_k)_{k\geq 0}$ tends to U. Moreover, Corollary 3.7 implies that $(T_{A_k})_{k\geq 0}$ tends to the identity, which implies that $(T_{A_k} P_k)_{k\geq 0}$ tends to U. On the other hand, Lemma 2.4 implies that P_k is periodic and A_k is a fundamental domain for it. Thus, $T_{A_k} P_k$ and T have the same orbits by Lemma 2.3, and the proof is complete. ∎

Let ERG denote the set of ergodic transformations in $\mathrm{Aut}(X,\mu)$.

PROPOSITION 4.18. *Let φ be a sublinear metric-compatible function and fix $T \in \mathrm{ERG}$. Then $\mathrm{ERG} \cap [T]_\varphi$ is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \mathsf{d}_{\varphi,T})$.*

Proof. By [Kec10, Thm. 3.6], $\mathrm{ERG} \cap [T]$ is a G_δ set in $(\mathrm{APER} \cap [T], d_u)$. Thus $\mathrm{ERG} \cap [T]_\varphi$ is a G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \mathsf{d}_{\varphi,T})$. Finally, since T is ergodic,

$$\mathrm{ERG} \cap [T]_\varphi \supseteq \{S \in \mathrm{APER} \cap [T]_\varphi \colon S \text{ and } T \text{ have the same orbits}\}.$$

Thus, Proposition 4.17 shows that $\mathrm{ERG}\cap[T]_\varphi$ is dense in $(\mathrm{APER}\cap[T]_\varphi, \mathsf{d}_{\varphi,T})$, and the proof is complete. ■

REMARK 4.19. The hypothesis that φ is sublinear is necessary, since for any ergodic $T \in \mathrm{Aut}(X,\mu)$, $\mathrm{ERG} \cap [T]_1$ is not dense in $\mathrm{APER} \cap [T]_1$. Indeed, one can define a continuous *index map* $I \colon [T]_1 \to \mathbb{Z}$ by integrating the cocycle (see [LM18, Cor. 4.20] for the fact that it takes values in $\mathbb{Z}$). Then $I(U) \neq 0$ for every ergodic $U \in [T]_1$: by [LM18, Prop. 4.13] every ergodic $U \in [T]_1$ is either almost positive or almost negative. Then combining [LM18, Prop. 4.17 and Prop. 3.4] shows that U has positive or negative index, so $I(U) \neq 0$. Finally, there are aperiodic elements with index 0: if $A \subseteq X$ is measurable with $0 < \mu(A) < 1$, then the aperiodic transformation $U := T_A T_{X\setminus A}^{-1}$ has index zero (using again [LM18, Prop. 3.4]). By continuity of the discrete-valued index map, we conclude that $\mathrm{ERG} \cap [T]_1$ cannot be dense in $\mathrm{APER} \cap [T]_1$.

DEFINITION 4.20. A transformation $S \in \mathrm{Aut}(X,\mu)$ is *weakly mixing* if for all finite subsets $\mathcal{F} \subseteq \mathrm{MAlg}(X,\mu)$ and all $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that for all $A, B \in \mathcal{F}$,

$$|\mu(V^n(A) \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

Given a measurable subset $A \subseteq X$ of positive measure we will denote by μ_A the probability measure on A defined by $\mu_A(B) := \mu(A \cap B)/\mu(A)$. We say that a transformation $T \in \mathrm{Aut}(X,\mu)$ is *weakly mixing on A* if $T(A) = A$, and the restriction of T to A is weakly mixing as an element of $\mathrm{Aut}(A,\mu_A)$. The following result will be crucial in the proof of Theorem 4.15.

THEOREM 4.21 (Conze [Con72]). *Let $T \in \mathrm{ERG}$. Then the set*

$$\{A \in \mathrm{MAlg}(X,\mu) \colon T_A \text{ is weakly mixing on } A\}$$

is a dense G_δ set in $(\mathrm{MAlg}(X,\mu), d_\mu)$ where $d_\mu(A,B) := \mu(A \triangle B)$.

Denote by WMIX the set of weakly mixing transformations of $\mathrm{Aut}(X,\mu)$. We are finally ready to prove Theorem 4.15, which can be reformulated as follows.

THEOREM 4.22. *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{ERG}$. Then the set*

$$\{U \in \mathrm{WMIX} \cap [T]_\varphi \colon T \text{ and } U \text{ have the same orbits}\}$$

is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \mathsf{d}_{\varphi,T})$.

Proof. By the Baire category theorem, the intersection of two dense G_δ subsets is a dense G_δ subset. Hence by Proposition 4.17, it suffices to show that $\mathrm{WMIX} \cap [T]_\varphi$ is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \mathsf{d}_{\varphi,T})$, which will occupy the remainder of the proof.

By definition, a transformation U is weakly mixing if and only if for all finite subsets $\mathcal{F} \subseteq \mathrm{MAlg}(X,\mu)$ and all $\varepsilon > 0$, it belongs to the set

$$O_{\mathcal{F},\varepsilon} := \{V \in \mathrm{Aut}(X,\mu) \colon \exists n \in \mathbb{Z}, \forall A, B \in \mathcal{F}, |\mu(V^n(A) \cap B) - \mu(A)\mu(B)| < \varepsilon\}.$$

Observe that each $O_{\mathcal{F},\varepsilon}$ is open in $(\mathrm{Aut}(X,\mu), d_u)$. As before, denote by d_μ the metric on $\mathrm{MAlg}(X,\mu)$ defined by $d_\mu(A,B) = \mu(A \triangle B)$.

CLAIM 1. *Let $\mathcal{F} = \{A_1, \dots, A_n\}$ and $\mathcal{F}' = \{A_1', \dots, A_n'\}$ be subsets of $\mathrm{MAlg}(X,\mu)$. Fix $\varepsilon > 0$. If for every $i \in \{1, \dots, n\}$ one has $\mu(A_i \triangle A_i') < \varepsilon$, then*

$$O_{\mathcal{F},\varepsilon} \subseteq O_{\mathcal{F}',5\varepsilon}.$$

Proof of Claim. Let $V \in O_{\mathcal{F},\varepsilon}$. Fix $n \in \mathbb{Z}$ such that for all $i, j \in \{1, \dots, n\}$ we have $|\mu(V^n(A_i) \cap A_j) - \mu(A_i)\mu(A_j)| < \varepsilon$. We first remark that for every measurable $B \subset X$ and $i \in \{1, \dots, n\}$, we have $|\mu(B)\mu(A_i) - \mu(B)\mu(A_i')| < \varepsilon$ and $|\mu(B \cap A_i') - \mu(B \cap A_i)| < \varepsilon$. The result now follows from the triangle inequality and the fact that V preserves μ:

$$\begin{aligned}
|\mu(V^n(A_i') \cap A_j') - \mu(A_i')\mu(A_j')| &< |\mu(V^n(A_i') \cap A_j') - \mu(A_i')\mu(A_j)| + \varepsilon \\
&< |\mu(V^n(A_i') \cap A_j') - \mu(A_i)\mu(A_j)| + 2\varepsilon \\
&< |\mu(V^n(A_i') \cap A_j) - \mu(A_i)\mu(A_j)| + 3\varepsilon \\
&= |\mu(A_i' \cap V^{-n}(A_j)) - \mu(A_i)\mu(A_j)| + 3\varepsilon \\
&< |\mu(A_i \cap V^{-n}(A_j)) - \mu(A_i)\mu(A_j)| + 4\varepsilon \\
&= |\mu(V^n(A_i) \cap A_j) - \mu(A_i)\mu(A_j)| + 4\varepsilon \\
&< 5\varepsilon. \quad \blacksquare_{\text{Claim}}
\end{aligned}$$

Since (X,μ) is standard, we can fix a countable dense subset $\mathcal{M}$ of $(\mathrm{MAlg}(X,\mu), d_\mu)$. It follows from Claim 1 that

$$\mathrm{WMIX} = \bigcap_{\varepsilon \in \mathbb{Q}_+^*} \bigcap_{\mathcal{F} \subseteq \mathcal{M} \text{ finite}} O_{\mathcal{F},\varepsilon}. \tag{4.3}$$

In particular, WMIX is a G_δ set in $(\mathrm{Aut}(X,\mu), d_u)$ and hence $\mathrm{WMIX} \cap [T]_\varphi$ is a G_δ set in $([T]_\varphi, \mathsf{d}_{\varphi,T})$.

We now prove that WMIX is dense. By the Baire category theorem, it is enough to show that each $O_{\mathcal{F},\varepsilon}$ is dense in $(\mathrm{APER} \cap [T]_\varphi, \mathsf{d}_{\varphi,T})$. By Proposition 4.18 the set $\mathrm{ERG} \cap [T]_\varphi$ is dense in $\mathrm{APER} \cap [T]_\varphi$, so it is enough to prove that

$$\mathrm{ERG} \cap [T]_\varphi \subseteq \overline{O_{\mathcal{F},\varepsilon} \cap \mathrm{APER} \cap [T]_\varphi}. \tag{4.4}$$

So let us fix a finite subset $\mathcal{F} \subseteq \mathrm{MAlg}(X,\mu)$, a positive real ε and $U \in \mathrm{ERG} \cap [T]_\varphi$.

We let $(X_k)_{k\geq 0}$ be a sequence of measurable subsets such that $\mu(X_k) = 1 - 2^{-k}$. For all $k \geq 0$, we apply Conze's theorem to the transformation U_{X_k}, which is ergodic on X_k, to find a measurable subset $Y_k \subseteq X_k$ such that $\mu(Y_k) > 1 - 2^{-k+1}$ and U_{Y_k} is weakly mixing on Y_k. Set $V_k := U_{Y_k} T_{X \setminus Y_k}$. We claim that $(V_k)_{k\geq 0}$ tends to U. Indeed, since $\lim_k \mu(Y_k) = 1$, Proposition 4.12 shows that U_{Y_k} tends to U, while Corollary 3.7 shows that $T_{X\setminus Y_k}$ tends to the identity.

Claim 2. *For k large enough, we have $V_k \in O_{\mathcal{F},\varepsilon}$.*

Proof of Claim. For all $k \geq 0$, put $\mathcal{F}_k := \{A \cap Y_k : A \in \mathcal{F}\}$. Since U_{Y_k} is weakly mixing on Y_k, we have $U_{Y_k} \in O_{\mathcal{F}_k,\varepsilon/5}$. By construction, the transformations U_{Y_k} and V_k coincide on Y_k, so we also have $V_k \in O_{\mathcal{F}_k,\varepsilon/5}$. Since $\lim_k \mu(Y_k) = 1$, for k large enough and all $A \in \mathcal{F}$, we have $\mu(A \triangle (A \cap Y_k)) < \varepsilon/5$. We thus get $V_k \in O_{\mathcal{F},\varepsilon}$ for k large enough by Claim 1. $\blacksquare_{\text{Claim}}$

It follows immediately from Claim 2 that any ergodic element in $[T]_\varphi$ is a limit of aperiodic elements in $O_{\mathcal{F},\varepsilon} \cap [T]_\varphi$. This shows the inclusion (4.4), ending the proof of the theorem. $\blacksquare$

Appendix. Proof of Belinskaya's theorem. In this appendix, we present a short proof of Belinskaya's theorem due to Katznelson which is not publicly available to our knowledge [Kat80]. As in Belinskaya's original proof, a key step is the following theorem, of independent interest. To lighten notation, given a point $x \in X$, a map $T : X \to X$, and a subset $I \subseteq \mathbb{Z}$, we will write

$$T^I(x) := \{T^i(x) : i \in I\}.$$

Theorem A.1 (Katznelson). *Let T be an aperiodic measure-preserving transformation, suppose $U \in \mathrm{Aut}(X,\mu)$ has the same orbits as T and that for almost every $x \in X$, the symmetric difference of the respective positive T and U orbits, $T^{\mathbb{N}}(x) \triangle U^{\mathbb{N}}(x)$, is finite. Then T and U are conjugate.*

Proof. We will explicitly define an element S in $[T]$ such that $U = S^{-1}TS$. This will be done thanks to the following claim.

Claim. *For almost every $x \in X$, there exists a unique $j(x) \in \mathbb{Z}$ such that*

$$|T^{\mathbb{N}+j(x)}(x) \setminus U^{\mathbb{N}}(x)| = |U^{\mathbb{N}}(x) \setminus T^{\mathbb{N}+j(x)}(x)|.$$

Proof of Claim. For almost every $x \in X$, define $\tau_x : \mathbb{Z} \to \mathbb{Z}$ by

$$\tau_x(j) := |T^{\mathbb{N}+j}(x) \setminus U^{\mathbb{N}}(x)| - |U^{\mathbb{N}}(x) \setminus T^{\mathbb{N}+j}(x)|.$$

By assumption, for almost every x, the value $\tau_x(j)$ is finite for all $j \in \mathbb{Z}$. By considering the two cases $T^j(x) \in U^{\mathbb{N}}(x)$ and $T^j(x) \notin U^{\mathbb{N}}(x)$, we see that $\tau_x(j+1) = \tau_x(j) - 1$ for all $j \in \mathbb{Z}$. It follows that $\tau_x(j) = \tau_x(0) - j$ for all $j \in \mathbb{Z}$, so $j(x) := \tau_x(0)$ is the unique element we seek. $\blacksquare_{\text{Claim}}$

We set $S(x) := T^{j(x)}(x)$. By the above claim, $S(x)$ is the unique element of the T-orbit of x satisfying

$$|T^{\mathbb{N}}(S(x)) \setminus U^{\mathbb{N}}(x)| = |U^{\mathbb{N}}(x) \setminus T^{\mathbb{N}}(S(x))|. \tag{A.1}$$

By considering whether none, only one, or both of the points x and $S(x)$ belong to $T^{\mathbb{N}}(S(x)) \cap U^{\mathbb{N}}(x)$, we see that removing the point $S(x)$ from $T^{\mathbb{N}}(S(x))$ and the point x from $U^{\mathbb{N}}(x)$ does not perturb the above equation, so that

$$|T^{\mathbb{N}+1}(S(x)) \setminus U^{\mathbb{N}+1}(x)| = |U^{\mathbb{N}+1}(x) \setminus T^{\mathbb{N}+1}(S(x))|.$$

This can be rewritten as

$$|T^{\mathbb{N}}(TS(x)) \setminus U^{\mathbb{N}}(U(x))| = |U^{\mathbb{N}}(U(x)) \setminus T^{\mathbb{N}}(TS(x))|,$$

which by equation (A.1) yields the desired equivariance condition

$$SU(x) = TS(x).$$

We now have to check that $S \in [T]$. Using the fact that T and U are invertible and a straightforward induction, we find that $SU^n(x) = T^nS(x)$ for all $n \in \mathbb{Z}$. In particular, S induces a bijection from the T-orbit of x to the U-orbit of $S(x)$. Since $S(x) = T^{j(x)}(x)$ belongs to the T-orbit of x, which coincides with the U-orbit of x, we conclude that S induces a bijection on each T-orbit, in particular S is bijective. Finally, we check that S is measure-preserving. The sets $A_n := \{x \in X : S(x) = T^n(x)\}$ for $n \in \mathbb{Z}$ form a partition of X. If $B \subseteq X$ is measurable, we write $B = \bigsqcup_n A_n \cap B$ so that $\mu(S(B)) = \sum_n \mu(T^n(A_n \cap B)) = \sum_n \mu(A_n \cap B) = \mu(B)$. This ends the proof of Theorem A.1. $\blacksquare$

Given $T \in \mathrm{Aut}(X, \mu)$, denote by $\mathcal{R}_T \subseteq X \times X$ the equivalence relation whose classes are the T-orbits. Before proceeding with the proof of Belinskaya's theorem, we recall the following well-known lemma. Its usefulness towards proving Belinskaya's theorem was pointed out to us by Todor Tsankov.

Lemma A.2 (Mass-transport principle). *Let $T \in \mathrm{Aut}(X,\mu)$ and let $f\colon \mathcal{R}_T \to \mathbb{N}$ be a measurable map. Then*

$$\int_X \sum_{n\in\mathbb{Z}} f(x,T^n(x))\,d\mu = \int_X \sum_{n\in\mathbb{Z}} f(T^n(x),x)\,d\mu.$$

Proof. Since f is non-negative, Tonelli's theorem tells us that

$$\begin{aligned}\int_X \sum_{n\in\mathbb{Z}} f(x,T^n(x))\,d\mu &= \sum_{n\in\mathbb{Z}}\int_X f(x,T^n(x))\,d\mu = \sum_{n\in\mathbb{Z}}\int_X f(T^{-n}(x),x)\,d\mu\\ &= \sum_{n\in\mathbb{Z}}\int_X f(T^n(x),x)\,d\mu = \int_X \sum_{n\in\mathbb{Z}} f(T^n(x),x)\,d\mu. \qquad\blacksquare\end{aligned}$$

Theorem A.3 (Belinskaya's theorem). *Let $T \in \mathrm{Aut}(X,\mu)$ be ergodic, and let $U \in [T]_1$ with the same orbits as T. Then T and U are flip-conjugate.*

Proof. Define a T-invariant total order $\leq_T$ on each T-orbit by setting $x \leq_T y$ if there is $n \geq 0$ such that $y = T^n(x)$. We will write $x <_T y$ whenever $x \neq y$ and $x \leq_T y$. Define $f\colon \mathcal{R}_T \to \mathbb{N}$ by

$$f(x,y) := \begin{cases} 1 & \text{if } x \leq_T y <_T U(x) \text{ or } U(x) <_T y \leq_T x,\\ 0 & \text{otherwise.}\end{cases}$$

Let us denote by c_U the T-cocycle of U. By assumption, c_U is integrable. Note that $f(x,T^n(x)) = 1$ if and only if $0 \leq n < c_U(x)$ or $c_U(x) < n \leq 0$, so $\sum_{n\in\mathbb{Z}} f(x,T^n(x)) = |c_U(x)|$. We thus have

$$\int_X \sum_{n\in\mathbb{Z}} f(x,T^n(x))\,d\mu = \int_X |c_U(x)|\,d\mu < +\infty.$$

Using the mass-transport principle (Lemma A.2), we deduce that

$$\int_X \sum_{n\in\mathbb{Z}} f(T^n(x),x)\,d\mu < +\infty,$$

in particular for almost every $x \in X$, the sum $\sum_{n\in\mathbb{Z}} f(T^n(x),x)$ is finite.

This implies that for almost every $x \in X$, there are only finitely many integers n such that $U^n(x) \leq_T x <_T U^{n+1}(x)$ or $U^{n+1}(x) <_T x \leq_T U^n(x)$. Since the U-orbit of almost every point is infinite, for almost every x we must have either $\lim_{n\to+\infty} c_{U^n}(x) = +\infty$ or $\lim_{n\to+\infty} c_{U^n}(x) = -\infty$. By ergodicity of U and up to replacing U with its inverse, we can assume that for almost all $x \in X$ we have $\lim_{n\to+\infty} c_{U^n}(x) = +\infty$, in particular for all but finitely many $n \geq 0$, we have $U^n(x) \geq_T x$.

By the symmetric argument and the fact that T and U have the same orbits, for all but finitely many $n \leq 0$ we have $U^n(x) \leq_T x$, and therefore $\{T^n(x)\colon n \geq 0\} \mathbin{\triangle} \{U^n(x)\colon n \geq 0\}$ is finite. The conclusion now follows from Theorem A.1. $\blacksquare$

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