Topological rank for full groups of pmp equivalence relations

François Le Maître

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Probability measure preserving equivalence relations

Throughout the talk,

- (X, μ) denotes a standard probability space, e. g. $X = [0, 1]$ and μ is the Lebesgue measure.
- Γ denotes any countable discrete group.

Definition

Let $\Gamma \curvearrowright (X, \mu)$ be a pmp (probability measure preserving) action, the associated **pmp equivalence relation** $\mathcal{R}_{\Gamma \curvearrowright X}$ is the Borel subset of $X \times X$ defined by $(x, y) \in \mathcal{R}_{\Gamma \curvearrowright X}$ iff $y \in \Gamma \cdot x$.

Orbit equivalence

Definition

Let $\mathcal{R}$, $\mathcal{R}'$ be two pmp equivalence relations on (X,μ). They are **orbit equivalent** if there exists a pmp bijection $\varphi : (X,\mu) \to (X,\mu)$ such that for a.e. $x \in X$,

$$\varphi([x]_{\mathcal{R}}) = [\varphi(x)]_{\mathcal{R}'}.$$

If $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$ and $\mathcal{R}' = \mathcal{R}_{\Lambda \curvearrowright X}$, one also say that the two actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (X,\mu)$ are orbit equivalent.

Theorem (Orstein-Weiss 80)

Any two ergodic actions of any two amenable groups are orbit equivalent.

Theorem (Esptein-Ioana-Kechris-Tsankov 08)

If Γ is non amenable, then equivalence relation "being orbit equivalent" on its space of actions is not classifiable by countable structures.

Two metrisable group topologies on $\mathrm{Aut}(X, \mu)$

Let $\mathrm{Aut}(X, \mu)$ be the group of pmp bijections $T : (X, \mu) \to (X, \mu)$, identified if they agree a.e.
It carries two natural metrisable group topologies:

- The **weak** topology, defined by $T_n \to T$ iff for all Borel $A \subseteq X$,

  $$\mu(T_n(A) \triangle T(A)) \to 0.$$

  It is a Polish group topology.
- The **uniform** topology, induced by the **uniform metric** d_u defined by

  $$d_u(T, T') = \mu(\{x \in X : T(x) \neq T'(x)\}).$$

  d_u is complete but not separable ($\mathbb{S}_1 \curvearrowright (\mathbb{S}_1, Haar)$ by left translation is discrete).

Definition

Let $\mathcal{R}$ be a pmp equivalence relation on (X, μ). Its **full group**, denoted by $[\mathcal{R}]$, is

$$[\mathcal{R}] = \{T \in \mathrm{Aut}(X, \mu) : (x, T(x)) \in \mathcal{R} \text{ for all } x \in X\}.$$

Fact

Full groups are closed separable subgroups of $\mathrm{Aut}(X, \mu)$ *for the uniform topology.*

So these are Polish groups!

Let $\mathcal{R}, \mathcal{R}'$ be two pmp equivalence relations on (X, μ).
If $\varphi \in \mathrm{Aut}(X, \mu)$ is an orbit equivalence between $\mathcal{R}$ and $\mathcal{R}'$, then

$$\varphi[\mathcal{R}]\varphi^{-1} = [\mathcal{R}'].$$

So the full group, seen as a (topological) group up to (topological) group isomorphism, is an invariant of orbit equivalence.

Theorem (Dye's reconstruction theorem)

The full groups of pmp ergodic equivalence relations, as abstract groups seen up to abstract group isomorphism, are ***complete*** *invariants of orbit equivalence.*

Two examples

Let $\mathcal{R}$ be a pmp equivalence relation on (X, μ).

Theorem (Eigen 81)

$\mathcal{R}$ is ergodic iff $[\mathcal{R}]$ is simple.

Theorem (Giordano-Pestov 05)

$\mathcal{R}$ is amenable iff $([\mathcal{R}], d_u)$ is extremely amenable.

An invariant for topological groups: the topological rank

Definition

Let G be a topological group. Its **topological rank** $t(G)$ is defined by

$$t(G) = \inf\{n \in \mathbb{N} : \exists g_1, \ldots, g_n \in G \text{ such that } \overline{\langle g_1, \ldots, g_n \rangle} = G\}.$$

Examples

- $t(\mathbb{S}^1) = 1$, actually $t(\mathbb{T}^n) = 1$ for all $n \in \mathbb{N}$.
- $t(\mathbb{R}^n) = n + 1$.
- (Schreier-Ulam) If G is compact connected metrisable, then $t(G) \leq 2$.
- $t(\mathcal{U}(\ell^2)) = 2$.
- (Prasad) $t(\mathrm{Aut}(X, \mu)) = 2$.

Furthermore, in all the above examples, the set of all $(g_1, \ldots, g_{t(G)}) \in G^{t(G)}$ which generate a dense subgroup is a dense G_δ in $G^{t(G)}$.
Actually, this set is always a G_δ for G Polish.

A natural question

Question (Kechris)

What about $t([\mathcal{R}])$?

Observation

Let $T_1, ..., T_n \in [\mathcal{R}]$ be topological generators of $[\mathcal{R}]$, then they generate the equivalence relation $\mathcal{R}$.

So before understanding topological generators of the full group $[\mathcal{R}]$, we should understand the generators of the equivalence relation $\mathcal{R}$.

Define the **pseudo full group** of $\mathcal{R}$ to be the set

$$[[\mathcal{R}]] = \{\varphi : \mathrm{dom}\,\varphi \subseteq X \to \mathrm{rng}\,\varphi \subseteq X \text{ such that } \forall x \in A, (x, \varphi(x)) \in \mathcal{R}\}$$

Definition (Levitt 95)

A **graphing** of $\mathcal{R}$ is a countable subfamily $\Phi = (\varphi_i)_{i\in I}$ of $[[\mathcal{R}]]$. Its **cost** is defined by $\mathrm{Cost}\,\Phi = \sum_{i\in I} \mu(\mathrm{dom}\,(\varphi_i))$.

Definition

Say that a graphing $\Phi = (\varphi_i)_{i\in I}$ **generates** $\mathcal{R}$ if $\mathcal{R}$ is the smallest equivalence relation whose pseudo full group contains $\{\varphi_i\}_{i\in I}$. The cost of $\mathcal{R}$ is then defined by

$$\mathrm{Cost}(\mathcal{R}) = \inf\{\mathrm{Cost}\,\Phi : \Phi \text{ generates } \mathcal{R}\}$$

Theorem (Levitt 95)

Let $\mathcal{R}$ be a pmp ergodic equivalence relation induced by a $\mathbb{Z}$-action. Then $\mathrm{Cost}(\mathcal{R}) = 1$.

Theorem (Gaboriau 00)

Let $n \in \mathbb{N}$, and let $\mathbb{F}_n$ be the free group on n generators. Then any pmp ***free*** *action of $\mathbb{F}_n$ induces a pmp equivalence relation of cost n.*

So for $n \neq m$, free actions of free groups of ranks n and m can never be orbit equivalent!

Suppose that $T_1, \ldots, T_n$ are topological generators of $[\mathcal{R}]$. Then we must have $n \geq \mathrm{Cost}(\mathcal{R})$ by definition. Let us show that $n \neq \mathrm{Cost}(\mathcal{R})$.

Theorem (Gaboriau 00)

*Suppose that $n = \mathrm{Cost}(\mathcal{R})$ and $\Phi = (T_1, \ldots, T_n)$ generates $\mathcal{R}$. Then $T_1, \ldots, T_n$ induce a **free** action of $\mathbb{F}_n$.*

In particular, $T_1, \ldots, T_n$ generate a discrete subgroup of $[\mathcal{R}]$, hence closed, which is a contradiction. In the end, we have

$$t([\mathcal{R}]) \geq \lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1.$$

Theorem (Kittrell-Tsankov 08)

Let $\mathcal{R}$ be ergodic, then the following inequality holds

$$\lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1 \leq t([\mathcal{R}]) \leq 3(\lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1).$$

This was later refined by Matui in 2011, who showed that

$$t([\mathcal{R}]) \leq 2(\lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1).$$

Marks (unpublished) has also obtained that

$$t([\mathcal{R}_{\mathbb{F}_n \curvearrowright X}]) \leq 2n.$$

Note that these results yield that for $n \neq m$ sufficiently far apart, full groups induced by $\mathbb{F}_n$ and $\mathbb{F}_m$ actions can never be isomorphic for purely topological reasons.

The key theorems

Theorem (Kittrell-Tsankov 08)

Let $(\mathcal{R}_i)_{i\in I}$ be a countable family of pmp equivalence relations, let $\mathcal{R}$ be the smallest pmp equivalence relation containing all the $\mathcal{R}_i$'s. Then

$$[\mathcal{R}] = \overline{\left\langle \bigcup_{i\in I} [\mathcal{R}_i] \right\rangle}$$

Theorem (Matui 11)

Let $\mathcal{R}_0$ be the ergodic hyperfinite equivalence relation. Then $t([\mathcal{R}_0]) = 2$.

The theorems relating $t([\mathcal{R}])$ and $\mathrm{Cost}(\mathcal{R})$ are then proven by finding finitely many hyperfinite subequivalences relations which generate $\mathcal{R}$.

Main result

Theorem (LM 13)

Let $\mathcal{R}$ be a pmp ergodic equivalence relation on (X, μ). Then

$$t([\mathcal{R}]) = \lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1.$$

Let $\mathcal{R}$ be a pmp ergodic equivalence relation. Assume $\mathrm{Cost}(\mathcal{R}) < n+1$, we want to find $n+1$ topological generators for $[\mathcal{R}]$.

Theorem (Dye 59)

$\mathcal{R}$ contains a hyperfinite ergodic subequivalence relation $\mathcal{R}_0$.

We fix such an ergodic hyperfinite equivalence relation $\mathcal{R}_0 \subseteq \mathcal{R}$, and a graphing Φ_0 of cost 1 which generates $\mathcal{R}_0$.

Theorem (Gaboriau, lemme III.5, "co-cost")

Let $\mathcal{R}_0 \subseteq \mathcal{R}$ be ergodic equivalence relations, where $\mathcal{R}_0$ is ergodic. Then for all $\epsilon > 0$, there exists a graphing Φ of $\mathcal{R}$ whose cost is less than $\mathrm{Cost}(\mathcal{R}) - 1 + \epsilon$, and such that $\Phi_0 \vee \Phi$ generates $\mathcal{R}$.

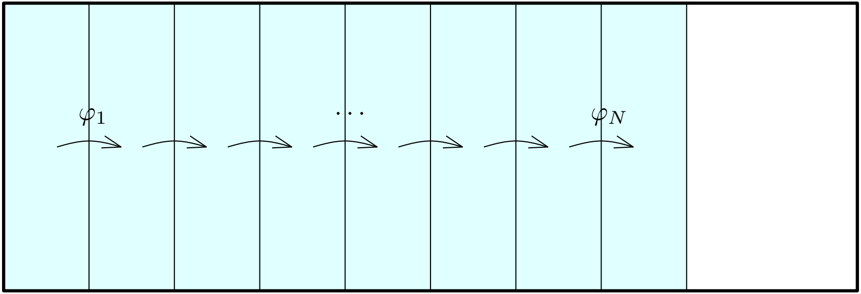
So if we let $n+1 = \lfloor Cost\, \mathcal{R} \rfloor$, we may fix Φ_1,..., Φ_n graphings, each of cost strictly less than 1, such that $\Phi_0 \vee \Phi_1 \vee \cdots \vee \Phi_n$ generates $\mathcal{R}$.

Working on Φ_i for $i = 1, ..., n$, which has cost < 1

Fact

Assume $A, B \subseteq X$ *are such that* $\mu(A) = \mu(B)$. *Then there exists* $\varphi \in [[\mathcal{R}_0]]$ *such that* $\operatorname{dom}\varphi = A$ *and* $\operatorname{rng}\varphi = B$.

By cutting, gluing elements of Φ_i and composing them with elements of $[[\mathcal{R}_0]]$, we may actually assume that $\Phi_i = (\varphi_j)_{j=1}^N$ looks like this:

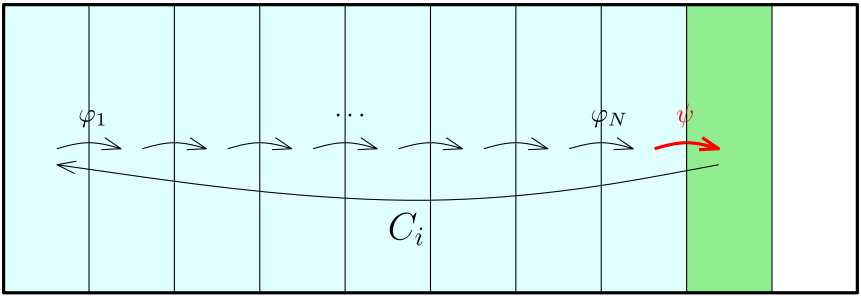


Working on Φ_i for $i = 1, \ldots, n$, which has cost < 1

If we choose N large enough, there is room for a green set of the same measure as each of the blue sets.

But now we may now add a $\psi \in [[\mathcal{R}_0]]$ to Φ_i whose domain is $\mathrm{rng}(\varphi_N)$, and whose range is the green set.

We can now "close" Φ_i and obtain a cycle $C_i \in [\mathcal{R}]$ generating $\mathcal{R}_{\Phi_i}$.



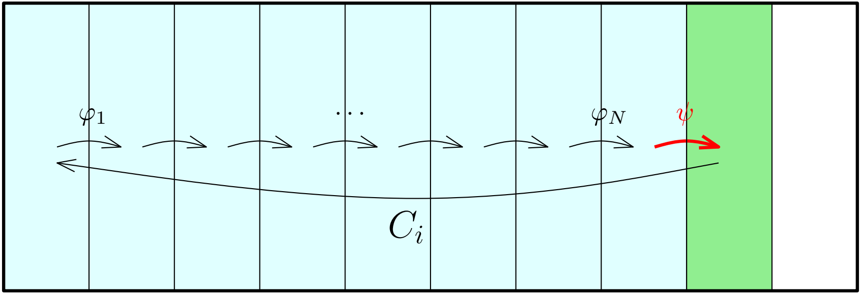
Working on Φ_i for $i = 1, ..., n$, which has cost < 1

Let T_0, U_0 be topological generators of $[\mathcal{R}_0]$.

Claim

The closed group generated by T_0, U_0 and C_i contains $[\mathcal{R}_i]$.

Using again [Kittrell-Tsankov], we get that $T_0, U_0, C_1, ..., C_n$ topologically generate $[\mathcal{R}]$. That's $n + 2$ topological generators, but we want $n + 1$!



A little trick

Fortunately, Matui's theorem is stronger than what was stated.

Theorem (Matui 11)

For all $\epsilon > 0$, there exists $T_0 \in [\mathcal{R}_0]$, and $U_0 \in [\mathcal{R}_0]$ such that $\{T_0, U_0\}$ topologically generates $[\mathcal{R}_0]$ and U_0 is an involution whose support has measure less than ϵ.

Then by ergodicity one can find such a $U_0 \in [\mathcal{R}_0]$ with support disjoint from the support of C_1. We may also assume that C_1 has odd orbits, i.e. N was odd. We now let $C_1' = C_1 U_0$.

Claim

$T_0, C_1', C_2, ..., C_n$ topologically generate $[\mathcal{R}]$.

Proof.

We have ${C_1'}^{N+2} = (C_1)^{N+2} U_0^{N+2} = U_0$, so that these elements generate a group containing $T_0, U_0, C_1, ..., C_n$. □

We have actually shown something stronger.

Theorem

Let $\mathcal{R}$ be a pmp ergodic equivalence relation. Then we have the formula

$$\mathrm{Cost}(\mathcal{R}) = \inf \left\{ \sum_{i=1}^{t([\mathcal{R}])} d_u(T_i, \mathrm{id}) : \overline{\left\langle T_1, \ldots, T_{t([\mathcal{R}])} \right\rangle} = [\mathcal{R}] \right\}.$$

What about the set of topological generators of $[\mathcal{R}]$?

First, because generic n tuples have arbitrarily small support, they cannot generate the whole equivalence relation, and so for $G = [\mathcal{R}]$, the set of $(g_1, ..., g_{t(G)}) \in G^{t(G)}$ generating a dense subgroup is not dense.

Theorem (LM)

Let T_0 be the odometer. Then $\{S \in [\mathcal{R}_{T_0}] : \overline{\langle S, T_0 \rangle} = [\mathcal{R}_{T_0}]\}$ is a dense G_δ.

Let *APER* denote the set of elements of $\mathrm{Aut}(X, \mu)$ having only infinite orbits.

Theorem (LM)

Let $\mathcal{R}$ be a cost one ergodic equivalence relation. Then

$$\{(T, S) \in (APER \cap [\mathcal{R}]) \times [\mathcal{R}] : \overline{\langle S, T \rangle} = [\mathcal{R}]$$

is a dense G_δ in $(APER \cap [\mathcal{R}]) \times [\mathcal{R}]$.

For $\mathcal{R}_0$ ergodic hyperfinite, the set $GEN(\mathcal{R}_0)$ of T's such that T generates $\mathcal{R}_0$ is a dense G_δ in $APER \cap [\mathcal{R}_0]$.

Theorem

Let $\mathcal{R}_0$ be the hyperfinite ergodic equivalence relation. Then

$$\{(T, S) \in GEN(\mathcal{R}_0) \times [\mathcal{R}_0] : \overline{\langle S, T\rangle} = [\mathcal{R}_0]$$

is a dense G_δ in $GEN(\mathcal{R}_0) \times [\mathcal{R}]$.

Question

Let T be an ergodic element of $\mathrm{Aut}(X, \mu)$. When is there $S \in [\mathcal{R}_T]$ such that $\overline{\langle T, S\rangle} = [\mathcal{R}_T]$? When is the set of such S's a dense G_δ?

Rank one transformations: yet another definition!

Definition

Let $T \in \mathrm{Aut}(X, \mu)$, $A \subseteq X$ and $N \in \mathbb{N}$. Suppose that for $A, T(A), ..., T^{N-1}(A)$ are all disjoint. Then we call the partition

$$\left(A, T(A), ..., T^{N-1}(A), X \setminus \bigsqcup_{i=0}^{N-1} T^i(A)\right)$$

a **Kakutani-Rohlin partition**.

Definition

$T \in \mathrm{Aut}(X, \mu)$ is **rank one** if for every finite partition $(B_1, ..., B_k)$ of X and $\epsilon > 0$, there exists a Kakutani-Rohlin partition $\mathcal{P}$ such that each B_i is ϵ-close in measure to a finite union of elements of $\mathcal{P}$.

Theorem (LM)

If T is a rank one transformation, then the set

$$\{S \in [\mathcal{R}_T] : \overline{\langle S, T\rangle} = [\mathcal{R}_T]\}$$

is a dense G_δ in $[\mathcal{R}_T]$.

Question

Is this true for all ergodic $T \in \mathrm{Aut}(X, \mu)$? In particular, is this true for Bernoulli shifts?