Around the commensurating full group

Conjugation ← too strong

quantitative OE ← in the middle

Orbit Equivalence ← too weak

Belinskaya '68: L^1 OE $\Rightarrow$ flip conjugate (either S, T are conjugate, or S, T^{-1} are)

Problem: only for pmp

Alt proof of B by Katznelson:

1) Show that if S, T are L^1 OE, then S "preserves" the positive T-orbits
2) conclude

(see [CJLMT23] for instance)

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1) Full groups etc

<u>Def</u> : X topological space is <u>Polish</u> if it is completely metrizable and second countable.

Ex : $\mathbb{R}$, $\mathrm{Sym}(\mathbb{Z})$, $\mathbb{N}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$ ...

<u>Def</u> A topological group is Polish if it is Polish

<u>Def</u> (X, μ) is standard if μ is a Borel atomless probability and X is Polish

All standard spaces are isomorphic (to $([0,1], \mathrm{Leb})$)

<u>Def</u> (X, μ) standard $\nu \sim \mu$ if ν is Borel and

$$\forall A, \ \nu(A) = 0 \Leftrightarrow \mu(A) = 0$$

$[\mu]$ is the <u>measure class</u> of μ

<u>Def</u> $\mathrm{Aut}(X, \mu) = \{T : X \to X \text{ bimeasurable such that } T_* \mu = \mu\} / \text{a.e.}$

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$\mathrm{Aut}(X,[\mu]) = \{T : X \to X \text{ ———————— } T_*\mu \sim \mu\} / \text{a.e.}$

Def For $T \in \mathrm{Aut}(X,[\mu])$, the <u>full group of T</u> is

$$[T] = \{S \in \mathrm{Aut}(X,[\mu]),\ \forall^* x,\ \exists\, c_S(x) \in \mathbb{Z} \text{ s.t. } S(x) = T^{c_S(x)}(x)\}$$

Def If T is aperiodic (i.e. all orbits are infinite) then c_S is unique, and is called the <u>cocycle of S</u>

Rem If $T \in \mathrm{Aut}(X,\mu)$, then $[T] < \mathrm{Aut}(X,\mu)$

Proof: $A_n = \{x \in X,\ c_S(x) = n\}$ $\quad X = \bigsqcup_{n\in\mathbb{Z}} A_n$

$$\mu(S^{-1}(B)) = \sum_{n\in\mathbb{Z}} \mu(S^{-1}(B \cap A_n)) = \sum_{n\in\mathbb{Z}} \mu(T^{-n}(B \cap A_n))$$

$$= \sum_{n\in\mathbb{Z}} \mu(B \cap A_n) = \mu(B)$$

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Prop: $[T]$ is Polish.

Def If T is aperiodic and pmp $[T]_1 = \{S \in [T] \text{ st } \int_X |c_S| \, d\mu < \infty\}$
[LM 18]

Belinskaya: If $S \in [T]_1$ and $T \in [S]_1$ ($S \in$ ergodic), then S, T are flip conjugate

2) Commensurating group $\quad (A \Delta B = A \setminus B \cup B \setminus A)$

Def $\mathrm{Sym}(\mathbb{Z}, \mathbb{N}) = \{\sigma \in \mathrm{Sym}(\mathbb{Z}),\ \mathcal{L}(\sigma) := |\mathbb{N} \Delta \sigma \mathbb{N}| < \infty\}$

$\mathcal{L}$ is called the length of σ

$\mathrm{Sym}(\mathbb{Z}, \mathbb{N}) = \mathrm{Homeo}^+(\mathbb{Z} \cup \{\pm\infty\})$

$\mathrm{Sym}(\mathbb{Z}, \mathbb{N}) \curvearrowright \mathrm{Comm}(\mathbb{N}) \quad (\{A,\ |A \Delta \mathbb{N}| < \infty\})$

$\mathrm{Sym}(\mathbb{Z}, \mathbb{N}) \hookrightarrow \mathrm{Sym}(\mathrm{Comm}\ \mathbb{N})$

We endow $\mathrm{Sym}(\mathbb{Z}, \mathbb{N})$ with the induced topology.

Prop: $\mathrm{Sym}(\mathbb{Z}, \mathbb{N})$ is Polish

Proof: $g \in \mathrm{Sym}(\mathrm{Comm}\ \mathbb{N})$, $A \in \mathbb{Z}$, define

$$g \cdot A = g(A \Delta \mathbb{N}) \Delta g(\mathbb{N})$$

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$$\mathrm{Sym}(\mathbb{Z}, \mathbb{N}) = \{ g \in \mathrm{Sym}(\mathrm{Comm}\,\mathbb{N}),\ |g\cdot A| = |A|\ \ g\cdot(A \cup B) = g\cdot A \cup g \cdot B,$$
$$\bigcup_n g \cdot [\![0, n]\!] = g(\mathbb{N}) \}$$

$\subset$: ok

$\supset$: $g \in RHS$, $|g \cdot \{k\}| =: \{\sigma(k)\}$, $\sigma \in \mathrm{Sym}\,\mathbb{Z}$

$$\forall A \Subset \mathbb{Z}, \quad g \cdot A = \sigma(A)$$
and $\bigcup_n g \cdot [\![0, n]\!] = \bigcup_n \sigma([\![1, n]\!]) = \sigma(\mathbb{N}) = g(\mathbb{N})$

$$g(A \Delta \mathbb{N}) = g \cdot A \Delta g(\mathbb{N}) = \sigma(A) \Delta \sigma(\mathbb{N}) = \sigma(A \Delta \mathbb{N}) \quad \blacksquare$$
$$\phi : \mathrm{Comm}(\mathbb{N}) \to \mathcal{P}_f(\mathbb{Z})$$
$$A \longmapsto A \Delta \mathbb{N}$$

Def Index map
$$I(\sigma) = |\mathbb{N} \setminus \sigma(\mathbb{N})| - |\sigma(\mathbb{N}) \setminus \mathbb{N}|$$

I is a continuous morphism to $\mathbb{Z}$

$I(\sigma)$ counts the number of orbits $-\infty \to +\infty$ $\left(\begin{array}{l} \sigma^n(x) \xrightarrow[-\infty]{} -\infty \\ \sigma^n(x) \xrightarrow[+\infty]{} +\infty \end{array} \right)$

$-$ the $\underline{\hspace{6em}}$ $+\infty \to -\infty$.

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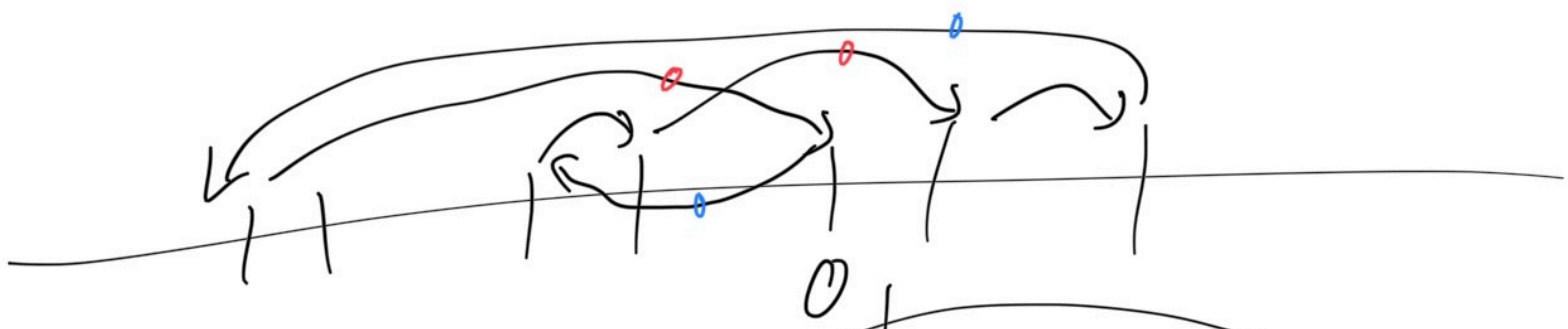

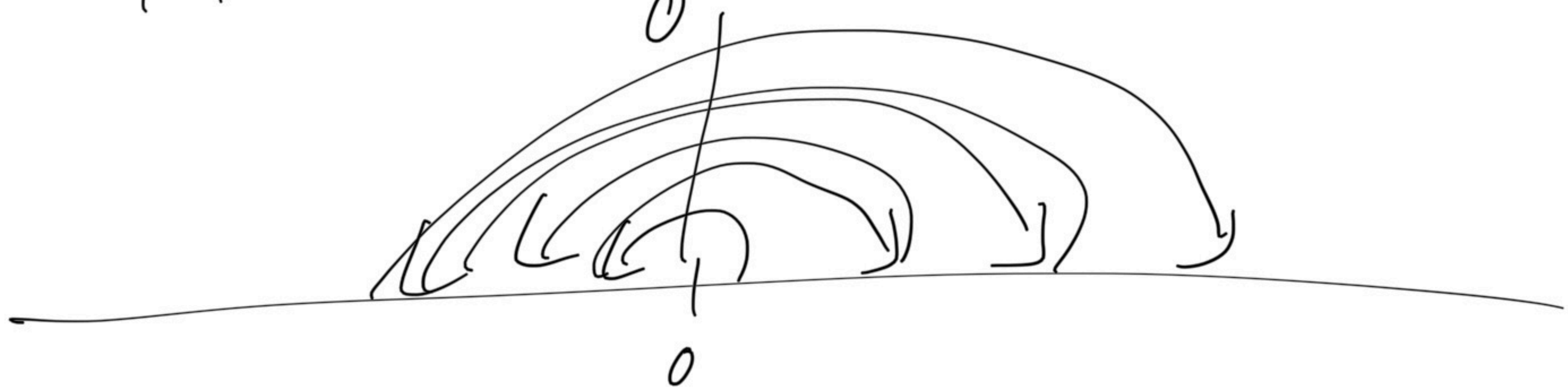

3) Commensurating full group!

Def (X,μ) standard, Y Polish,

$$L^0(X,\mu,Y) = \{ f : X \to Y \text{ measurable} \}/_{a.e.}$$

Define a topology on L^0 by $V_\varepsilon(f) = \{ g \in L^0,\ \mu(x\ d_Y(g(x), f(x)) < \varepsilon) > 1-\varepsilon \}$
where d_Y is a complete compatible metric.

Prop (Moore): This topology does not depend on d_Y and $\mu \in [\mu]$

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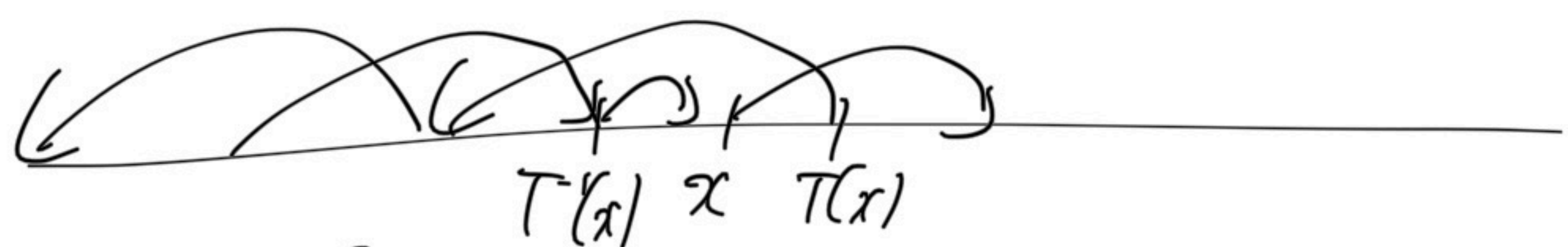
$$\phi : [T] \hookrightarrow L^0(X,\mu,\mathrm{Sym}(\mathbb{Z}))$$

$$S \longmapsto \phi(S)(x)(n) = n + c_S(T^n(x))$$

$$\left(= \phi(S)(T^n(x))(0)\right)$$

$S \circlearrowright T^{\mathbb{Z}}(x)$



$\phi(S)(x)$

Fact: this embedding is topological [KLM15, Prop 13]

<u>Def</u> $[T]_{com} = [T] \cap L^0(X,\mu\ \mathrm{Sym}(\mathbb{Z},\mathbb{N}))$, with the induced topology of $L^0(X,\mathrm{Sym}(\mathbb{Z},\mathbb{N}))$

<u>Lemma</u>: $[T]_{com}$ is Polish

Proof $[T]$ is closed in $L^0(X,\mu,\mathrm{Sym}(\mathbb{Z}))$ and the topology of $L^0(X,\mu,\mathrm{Sym}(\mathbb{Z},\mathbb{N}))$ is finer than that of $L^0(X,\mu,\mathrm{Sym}(\mathbb{Z}))$

($\sigma_n \longrightarrow 1$ in $\mathrm{Sym}(\mathbb{Z},\mathbb{N})$ iff $\sigma_n \longrightarrow 1$ in $\mathrm{Sym}(\mathbb{Z})$ and $\sigma_n(\mathbb{N}) =_* \mathbb{N}$ $n \gg 1$)

[Assumption T is ergodic]

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<u>Def</u> $I(\underbrace{\phi(S)(x)}_{\in \mathrm{Sym}(\mathbb{Z},\mathbb{N})}) = I(\phi(S)(T(x)))$ by ergodicity

$I(\phi(S)(x))$ does not depend on X: we define $I(S)$ to be this common value.

I is a continuous morphism $I : [T]_{com} \to \mathbb{Z}$.

If T is pmp, $S \in [T]_1$
$$I(S) = \int_X c_S(x)\, d\mu(x)$$

$[T]_1 < [T]_{com}$

A lot is known for $[T]_1$: complete invariant of flip conjugacy, quasi-isometry type, ker I is also very nice

Most results carry over to $[T]_{com}$.

Comes from the framework [LMS21]:

$G < \mathrm{Aut}(X, [\mu])$ is

- finitely full if $\forall X = \bigsqcup_{i=1}^{n} A_i$, $\forall g_i \in G$ st $X = \bigsqcup_{i=1}^{n} g_i A_i$, then
$$\coprod_{i=1}^{n} g_i|_{A_i} \in G$$

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- induction friendliness

for $A \subset X$, $T \in \mathrm{Aut}(X, [\mu])$, for $x \in A$ define $n_{A,T}(x) = \{\min k > 0, T^k(x) \in A\}$.

if it exists, $T_A(x) = \begin{cases} T^{n_{A,T}(x)}(x) & \text{if } x \in A \\ x & \text{otherwise} \end{cases}$ (so $\mathrm{supp}(T_A) \subset A$.)

G is I.F. if $\forall A \subset X$, $S \in G$, $S_A \in G$, and if $\overset{\uparrow}{\bigcup_{n \geq 0}} \underbrace{A_n}_{S\text{-invariant}} = A$,

$\lim_n S_{A_n} = S_A$

<u>Lemma</u> $S \in [T]_{con}$, $S = S_p S_+ S_-$ with disjoint supports,

S_p periodic,

S_+ almost positive: $c_{S_+^n}(x) \to \infty$ $\forall x \in \mathrm{supp}\, S$

S_- almost negative $c_{S_-^n}(x) \to -\infty$ $\forall^* x \in \mathrm{supp}\, S_-$

Proof

$A_p = \{x \in X$ st $S^{\mathbb{Z}}(x)$ is finite$\}$ $\qquad (S_p = S_{A_p})$

$A_+ = \{x \in X, c_{S_+^n}(x) \longrightarrow \infty\}$

$A_- = \text{———————} -\infty\}$

$A_p \cup A_+ \cup A_- = X$

Next time:

if S is positive ($c_S(x) \geq 0$), then

$$S = T_{A_1} T_{A_2} \cdots T_{A_k}$$

The following gps are equal:

G_{per}

$\overline{G_{inv}}$

$\overline{D([T_{com}])}$

$\ker I$