# Orbit equivalence and entropy for locally finite group: detailed proof of a result of Stepin

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**Abstract**

We give an exposition of the proof of Theorem 1 from [Ste71] which is more detailed than the original one.

Let $\Gamma = \bigoplus_{n\geqslant 1}\mathbb{Z}/2\mathbb{Z}$, denote by $\gamma_i$ the $i$'th coordinate natural generator and $\Gamma_i = \langle\gamma_1,\dots,\gamma_i\rangle = \bigoplus_{n=1}^{i}\mathbb{Z}/2\mathbb{Z}$.

**Theorem 1** (Stepin)**.** *Let $S, T : \Gamma \to \mathrm{Aut}(X,\mu)$ be two free pmp actions of $\Gamma$ on a standard probability space $(X,\mu)$, suppose that for every $n \in \mathbb{N}$, $S(\Gamma_n)$ has the same orbits as $T(\Gamma_n)$, then*

$$|h(S) - h(T)| \leqslant \log 2.$$

*Proof.* For brevity, we let $S_\gamma = S(\gamma)$ and $T_\gamma = T(\gamma)$ be the pmp bijections induced by these two actions. We are going to bound the entropy of $S$ in terms of that of $T$, which by symmetry will yield the desired result. The key to this bound is the following claim, which estimates how much larger the $S(\Gamma_n)$-saturation of a $T(\Gamma_n)$-invariant partition is.

**Claim.** Define by induction $p_1 = 1$ and $p_{k+1} = 2^k p_k^2$. Let $n \geqslant 1$, let $\mathcal{P}$ be a partition which is $T(\Gamma_n)$ invariant. Then $\bigvee_{\gamma\in\Gamma_n} S_\gamma\mathcal{P}$ is obtained by dividing each atom of $\mathcal{P}$ in at most $p_n$ elements.

*Proof of the claim.* The proof is by induction. For $n = 1$ we defined $p_1 = 1$ and our claim is clear since two involutions sharing the same orbits must be equal.

Suppose the claim has been proved at rank $n$, let $\mathcal{P}$ be $T(\Gamma_{n+1})$-invariant. In particular $\mathcal{P}$ is $T(\Gamma_n)$ invariant so by our induction hypothesis if we let $\mathcal{Q}$ be the partition obtained by $S(\Gamma_n)$-translating $\mathcal{P}$, $\mathcal{Q}$ is obtained by splitting each element of $\mathcal{P}$ into at most $p_n$ pieces. Observe that $\bigvee_{\gamma\in\Gamma_{n+1}} S_\gamma\mathcal{P} = \mathcal{Q}\vee S(\gamma_{n+1})\mathcal{Q}$ since $\gamma_{n+1}$ is an involution commuting with $\Gamma_n$ and $\Gamma_{n+1} = \langle\Gamma_n,\gamma_{n+1}\rangle$.

Observe that by freeness each $S(\Gamma_{n+1})$-orbit splits into two $S(\Gamma_n)$-orbits, and is equal to a $T(\Gamma_{n+1})$-orbit. Since $S(\Gamma_n)$ has the same orbits as $T(\Gamma_n)$, the cocycle of $S_{\gamma_{n+1}}$ must have its $n+1$ coordinate equal to 1. We thus get a partition $(A_\gamma)_{\gamma\in\Gamma_n}$ of $X$ such that for all $x \in A_\gamma$,

$$S_{\gamma_{n+1}}(x) = T_\gamma T_{\gamma_{n+1}}(x).$$

We claim that the partition

$$\tilde{\mathcal{Q}} = \{A_\gamma \cap C \cap T_\gamma T_{\gamma_{n+1}}(C') \colon C, C' \in \mathcal{Q}, \gamma \in \Gamma_n\}$$

refines $\mathcal{Q}$ while being $S(\gamma_{n+1})$-invariant. First note that each $A_\gamma$ is $S_{\gamma_{n+1}}$-invariant since it is the place where $S_{\gamma_{n+1}}$ coincides with another fixed involution, namely $T_\gamma T_{\gamma_{n+1}}$ . It follows from the definition of $A_\gamma$ that we can rewrite $\tilde{\mathcal{Q}}$ as

$$\tilde{\mathcal{Q}} = \{A_\gamma \cap C \cap S_{\gamma_{n+1}}(C') \colon C, C' \in \mathcal{Q}, \gamma \in \Gamma_n\}.$$

Since $S_{\gamma_{n+1}}$ is an involution, it then easily follows that $\tilde{Q}$ is $S_{\gamma_{n+1}}$-invariant.

By our assumption on $\mathcal{P}$, the pmp bijection $T_\gamma T_{\gamma_{n+1}}$ permutes the elements of $\mathcal{P}$, each which was cut into at most $p_n$ pieces of the form $C$ or $C'$ when obtaining $\mathcal{Q}$. Since $|\Gamma_n| = 2^n$, we get from the first definition of $\tilde{Q}$, namely

$$\tilde{Q} = \{A_\gamma \cap C \cap T_\gamma T_{\gamma_{n+1}}(C') \colon C, C' \in \mathcal{Q}, \gamma \in \Gamma_n\}$$

that $\tilde{Q}$ was obtained by cutting each element of $\mathcal{P}$ into at most $2^n \times p_n \times p_n$ pieces. As a consequence[1], each element of $\mathcal{P}$ will be split in at most $2^n \times p_n \times p_n = p_{n+1}$ pieces when first constructing $\mathcal{Q}$ and then constructing $\mathcal{Q} \vee \mathcal{S}_{\gamma_{n+1}} \mathcal{Q} = \bigvee_{\gamma \in \Gamma_{n+1}} S_\gamma \mathcal{P}$. This finishes the proof of the induction. $\square_{\text{claim}}$

We can now compare entropies. Given a finite set of pmp bijections $F$ and a partition $\mathcal{R}$, we denote by

$$\mathcal{R}^F = \bigvee_{T \in F} T(\mathcal{R})$$

the partition generated by the $T$-translates of $\mathcal{R}$ for $T \in F$. Fix a finite partition $\mathcal{R}$. Then by definition $h(T(\Gamma), \mathcal{R}) = \lim_{n \to +\infty} \frac{H(\mathcal{R}^{T(\Gamma_n)})}{2^n}$.

Now let $\mathcal{Q}_n = (\mathcal{R}^{T(\Gamma_n)})^{S(\Gamma_n)}$, which refines $\mathcal{R}^{S(\Gamma_n)}$. By our previous claim applied to $\mathcal{P} = \mathcal{R}^{T(\Gamma_n)}$,

$$H(\mathcal{Q}_n) \leqslant H(\mathcal{P}^{T(\Gamma_n)}) + \log p_n$$

(this follows by conditioning on the elements of $\mathcal{P}^{T(\Gamma_n)}$ since these have been split in at most $p_n$ pieces and a partition in $p_n$ pieces has entropy at most $\log p_n$). In particular $\frac{H(\mathcal{Q}_n)}{2^n} \leqslant \frac{\log p_n}{2^n}$. Now $\log p_{n+1} = 2 \log p_n + n \log 2$ so

$$\frac{\log p_{n+1}}{2^{n+1}} = \frac{\log p_n}{2^n} + \frac{n \log 2}{2^{n+1}}.$$

Since the series $\sum_n \frac{n \log 2}{2^{n+1}}$ converges to $\log 2$, we conclude that

$$h(S(\Gamma), \mathcal{R}) \leqslant h(T(\Gamma), \mathcal{R}) + \log 2 \leqslant h(T) + \log 2$$

Taking a supremum over all $\mathcal{R}$ we get

$$h(S) \leqslant h(T) + \log 2,$$

and the result follows by symmetry. $\square$

[1]Note that here we do not claim (and it might very well not be the case) that $\tilde{Q}$ is $S(\Gamma_{n+1})$-invariant, we only need that it is $S(\gamma_{n+1})$-invariant.

**Remark 2.** In particular, the above theorem says that if we restrict orbit equivalence of $\Gamma$-actions by requiring it to take $\Gamma_n$-orbits to $\Gamma_n$-orbits for every $n \in \mathbb{N}$, we get countably many different actions up to this finer equivalence relation on actions. We should also mention that Stepin refines the above result and gets continuum many such actions in [Ste71, Theorem 2], which uses Theorem 1. Vershik obtained a similar result to Stepin's in the same journal issue as Stepin [Ver71]. Moreover, another result of Vershik shows that given $S, T : \Gamma \to \mathrm{Aut}(X, \mu)$, there is a sequence $(n_k)$ and an orbit equivalence which takes $S(\Gamma_{n_k})$ orbits to $T(\Gamma_{n_k})$-orbits for every $k \in \mathbb{N}$ [Ver68]. The above result shows that in general, one cannot take $n_k = k$.

# References


[Ste71] A. M. Stepin. On entropy invariants of decreasing sequences of measurable partitions. *Functional Analysis and Its Applications*, 5(3):237–240, 1971.

[Ver68] A. M. Vershik. Theorem on lacunary isomorphisms of monotonic sequences of partitions. *Functional Analysis and Its Applications*, 2(3):200–203, 1968.

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