Lectures on masas

François Le Maître

Contents

1 Definition and first examples from countable groups

Let (M, τ) be a finite separable von Neumann algebra. A subalgebra $A \subseteq M$ is called maximal abelian if it is maximal as an abelian subalgebra. This means that there is no $x \in M \setminus A$ which commutes with A.

In this section, our only source of examples will be von Neumann algebras of countable infinite groups. So let us fix a countable infinite group Γ. Recall that each element of $L\Gamma$ has a unique Fourier decomposition $x = \sum_\gamma a_\gamma u_\gamma$, where the convergence is in the 2-norm (but not in any of the usual topologies on $\mathcal{B}(\mathcal{H})$!) and $a_\gamma = \langle x\delta_e, \delta_\gamma\rangle = \langle x\delta_e, u_\gamma\delta_e\rangle = \tau(u_\gamma^* x) = \tau(x u_\gamma^*)$. Let us start by recalling when $L\Gamma$ is a factor.

Proposition 1.1. *The von Neumann algebra $L\Gamma$ is a factor iff Γ is i.c.c. (all the conjugacy classes of non trivial elements are infinite).*

Proof. Suppose $\gamma \in \Gamma$ has a finite conjugacy class F. Then by considering the average of the elements in the conjugacy class, we get an element of $L\Gamma$ which is invariant under conjugacy by unitaries from γ. In other words, our algebra has nontrivial center.

Conversely let $x = \sum_\gamma a_\gamma u_\gamma$ belong to the center. Then for all $h \in \Gamma$, we have $a_\gamma = a_{h\gamma h^{-1}}$. Indeed,

$$\begin{aligned}
\tau(x\lambda^*_{h\gamma h^{-1}}) &= \tau(x\lambda_h\lambda_\gamma^*\lambda_h^*)\\
&= \tau(\lambda_h^* x\lambda_h\lambda_\gamma^*)\\
&= \tau(x\lambda_h^*\lambda_h\lambda_\gamma^*)\\
&= \tau(x\lambda_\gamma^*)
\end{aligned}$$

Note that we used the trace property to go from the first to the second line. But the integrability of the squares of a_γ then forces that if a coefficient other than a_e is non zero, the conjugacy class of the corresponding element is finite. $\square$

More generaly, we have the following computation of relative commutants of subgroups of Γ.

Proposition 1.2. *Let Λ be a subgroup of Γ. Then the elements of $\Lambda' \cap L\Gamma$ are those whose Fourier coefficients are constant on each Λ-conjugacy class.* $\square$

When M is the von Neumann algebra of a group Γ and Λ is an abelian subgroup of Γ, one can consider $L\Lambda$ as an abelian subalgebra of $L\Gamma$. We first figure out when it is maximal. Clearly a sufficient condition is that Λ is a maximal subgroup of Γ. Moreover if $x \in L\Gamma$ commutes with all the elements of Λ, by the previous proposition its Fourier coefficients must be constant on Λ conjugates, so by the same proof as above we have:

Proposition 1.3. *$L\Lambda$ is a masa in $L\Gamma$ iff Λ is abelian and every element of $\Gamma \setminus \Lambda$ has an infinite Λ-conjugacy class.* $\square$

Note that a masa in a finite II_1 factor has to be diffuse, so all masas in separable von Neumann algebras are isomorphic as von Neumann algebras. However, the way they sit in the ambient algebra can be drastically different. Here is a first basic invariant introduced by Dixmier.

Definition 1.4. Let N be a subalgebra of a von Neuman algebra M. The normalizing algebra of N in M, denoted by $\mathcal{N}_M(N)$, is the von Neumann algebra generated by the unitaries u of M satisfying $uNu^* = N$. Say that N is

- Regular (or Cartan) if $\mathcal{N}_M(N) = M$;
- Semi-regular if $\mathcal{N}_M(N)$ is a factor;
- Singular if $\mathcal{N}_M(N) = N$.

Note that it can happen that a masa falls in none of the above categories. Such examples will be obtained later on via ergodic theory.

1.1 Regular masas

The following proposition is not hard to show.

Proposition 1.5. *Let Γ be a countable group, let $\Lambda \leqslant \Gamma$ be an abelian normal subgroup such that every element of $\Gamma \setminus \Lambda$ has an infinite Λ-conjugacy class. Then $L\Lambda$ is a Cartan subalgebra of $L\Gamma$.* $\square$

Example 1.6. Consider the group Γ whose elements are 2-by-2 matrices of the form $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ where $a \in \mathbb{Q}^*$ and $x \in \mathbb{Q}$. We have the following conjugation:

$$\begin{pmatrix} b & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} ab & bx+y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b^{-1} & -b^{-1}y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bx+(1-a)y \\ 0 & 1 \end{pmatrix}$$

So if $x \neq 0$ the conjugacy class of $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ is infinite, and if $a \neq 1$ it is also infinite, so our group Γ is i.c.c.

The subgroup Λ consisting of those elements of the form $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ is a normal subgroup. Every element of $\Gamma \setminus \Lambda$ is of the form $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ with $a \neq 1$, and its conjugate by $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ is equal to:

$$\begin{pmatrix} a & x + (1-a)y \\ 0 & 1 \end{pmatrix}$$

so its Λ-conjugacy class is infinite. So $L\Lambda$ is a Cartan subalgebra of $L\Gamma$.

In order to give a more elaborate example, let us first recall the definition of the semi-direct product of groups: given a group Γ acting on a group Λ by automorphism, the semi-direct product $\Lambda \rtimes \Gamma$ is the set $\Lambda \times \Gamma$ equipped with the product

$$(\lambda_1, \gamma_1)(\lambda_2, \gamma_2) = (\lambda_1(\gamma_1 \cdot \lambda_2), \gamma_1\gamma_2).$$

We view both Λ and Γ as subgroups of $\Lambda \rtimes \Gamma$ via the respective group embeddings $\lambda \mapsto (\lambda, 1_\Gamma)$ and $\gamma \mapsto (1_\Lambda, \gamma)$. Note that we then have, for all $g, \gamma \in \Gamma$ and all $\lambda \in \Lambda$,

$$g(\lambda, \gamma)g^{-1} = (g \cdot \lambda, g\gamma g^{-1}),$$

in particular $g\lambda g^{-1} = g \cdot \lambda$, using our previous identifications. It is also useful to give a description of the conjugacy action of Λ on the whole group: given $h \in \Lambda$, we have

$$h(\lambda, \gamma)h^{-1} = (h\lambda(\gamma \cdot h^{-1}), \gamma).$$

So when Λ is abelian, the Λ-stabilizer of (λ, γ) for the conjugacy action is the set of all $h \in \Lambda$ such that $h - \gamma \cdot h = 0$, so it is the subgroup $\mathrm{Fix}(\gamma)$. So the conjugacy class is infinite if and only if $\mathrm{Fix}(\gamma)$ has infinite index in Λ.

Exercise 1.1. Show that the previous example is isomorphic to the semi-direct product $\mathbb{Q} \rtimes \mathbb{Q}^*$, where $\mathbb{Q}^*$ acts by multiplication on $\mathbb{Q}$.

Example 1.7. Let us show that $L\mathbb{Z}^2 \leqslant L\mathbb{Z}^2 \rtimes Sl_2(\mathbb{Z})$ is Cartan. First, let us see why $\mathbb{Z}^2 \rtimes Sl_2(\mathbb{Z})$ is i.c.c. Clearly the $Sl_2(\mathbb{Z})$-action on $\mathbb{Z}^2$ has infinite orbits except for $(0,0)$. So it suffices to show that every element $\neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $Sl_2(\mathbb{Z})$ has an infinite conjugacy class in $Sl_2(\mathbb{Z})$, and then to show that the elements $((a,b), \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix})$ have an infinite conjugacy class. For a given matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its respective conjugates by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (which both belong to $Sl_2(\mathbb{Z})$) are equal to

$$\begin{pmatrix} a+c & (b+d)-(a+c) \\ c & d-c \end{pmatrix} \text{ and } \begin{pmatrix} a-b & b \\ (a+c)-(b+d) & b+d \end{pmatrix}$$

So we see that as soon as one of the coefficients c or b is non zero, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an infinite $Sl_2(\mathbb{Z})$ conjugacy class. Since the only 2 matrices the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ belonging to $Sl_2(\mathbb{Z})$ are $\pm\mathrm{id}$, we now only need to show that $((a,b), \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix})$ has an infinite conjugacy class. But if we conjugate it by an element $((u,v),\mathrm{id})$, we get

$$((u,v),\mathrm{id})\left((a,b),\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)((-u,-v),\mathrm{id}) = \left((a+2u,b+2v),\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

So our group is icc as wanted.

Now by construction of the semi-direct product, the subgroups $\mathbb{Z}^2$ is normal, so to finish the proof we need to show that every element of $\mathbb{Z}^2 \rtimes Sl_2(\mathbb{Z})$ which does not belong to $\mathbb{Z}^2$ has an infinite $\mathbb{Z}^2$ conjugacy class. Fix a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $Sl_2(\mathbb{Z})$ different from the identity, by our analysis on semi-direct products and the fact that Λ is abelian it suffices to show that the set of all $h-(\gamma\cdot h)$ is infinite when h ranges over $\mathbb{Z}^2$. So write $h = \begin{pmatrix} u \\ v \end{pmatrix}$, then

$$h - \gamma\cdot h = \begin{pmatrix} u-(au+bv) \\ v-(cu+dv) \end{pmatrix}$$

Since $\gamma \neq 1$, either $b \neq 0$ in which case by changing v we see that the first coordinate of the above vector can take infinitely many different values, or $c \neq 0$ in which case the second coordinate of the above vector can take infinitely many different values.

1.2 Singular masas

Consider the following condition on an abelian subgroup $H \leqslant G$: for all $B \Subset G\setminus H$, there is $h \in H$ such that $BhB \cap H = \emptyset$.

First note that this implies that LH is a masa. Indeed, take $g \in G \setminus H$, suppose for a contradiction that the conjugacy class g^H is finite. Then $(\gamma^{-1})^\Lambda$ is also finite, so there is $\lambda_0 \in \Lambda$ such that for all $\lambda_1, \lambda_2 \in \Lambda$,

$$\lambda_1\gamma\lambda_1^{-1}\lambda_0\lambda_2\gamma^{-1}\lambda_2^{-1} \notin \Lambda$$

But if we take $\lambda_1 = \lambda_0$ and $\lambda_2 = e$, this becomes $\lambda_0\gamma\gamma^{-1} = \lambda_0 \in \Lambda$, a contradiction.

Now let us see why this implies that LH is moreover singular. To this end, let u be a unitary in LG which normalizes LH. By the Kaplansky density theorem, we find a sequence of elements $t_n \in \mathbb{C}G$ such that $t_n \to u$ *-strongly and $\|t_n\| \leqslant 1$. Write $t_n = x_n + y_n$, where $x_n \in \mathbb{C}H$ and $y_n \in \mathbb{C}G \setminus H$. Note that $\|x_n\| \leqslant 1$ because x_n is the conditional expectation of t_n on LH.

Writing the union of the support of y_n and y_n^* as B_n, we find h_n such that $B_n h_n B_n \cap H = \emptyset$. Then the elements $y_n\lambda_{h_n}y_n^*$, $y_n\lambda_{h_n}x_n^*$ and $x_n\lambda_{h_n}y_n^*$ are all supported outside of H, in particular the conditional expectation of $t_n\lambda_{h_n}t_n^*$ onto LH is equal to $x_n\lambda_{h_n}x_n^*$.

But by the continuity of multiplication on bounded sets and the continuity of the conditional expectation, we then have that the 2 norm of the conditional expectation of $t_n\lambda_{h_n}t_n^*$ converges to the 2-norm of $u\lambda_{h_n}u*$ which is equal to 1. So the 2-norm of $x_n\lambda_{h_n}x_n^*$ goes to 1, and since it is bounded above by $\|x_n\|\,\|\lambda_{h_n}\|\,\|x_n\|_2$, we have $\|x_n\|_2 \to 1$, which means that $\|y_n\|_2 \to 0$. So u is a $\|\cdot\|_2$ limit of elements of $(LH)_1$, in particular it belongs to LH as wanted.

Example 1.8. Let us go back to the $\mathbb{Q} \rtimes \mathbb{Q}^*$ example, this time consider the group $H = \mathbb{Q}^*$, we show it satisfies the previous condition. Suppose we are given a family B of elements $(a_1, b_1), ..., (a_n, b_n)$ with all the a_i's non zero. Observe that

$$(a, b)(0, c)(a', b') = (a, bc)(a', b') = (a + bca', bcb')$$

so c is as wanted as soon as $a_i + b_j c a_j \neq 0$ for all i, j which we can always ensure.

Example 1.9. In $L\mathbb{F}_2 = \langle a, b\rangle$, consider the group generated by a. It is abelian, and easily seen to satisfy the singularity condition by taking a large enough power of a such that for any $b_1, b_2 \in B$, when reducing $b_1 a^k b_2$, one letter of a^k is kept in the middle, ensuring there is no simplification of a b letter.

1.3 Semi-regular masas

In order to obtain semi-regular masas, we need an extra intermediate subgroup. Suppose we have $H \leqslant K \leqslant G$, where LH is already a masa in LG, that both K and G are icc and that H is normal in G.

We introduce two conditions which are both sufficient for LH to be semi-regular, and which will actually ensure that its normalizing algebra is equal to LG.

(i) For all $k \in K \setminus G$, all $B \Subset K$, there is $h \in H$ such that $khk^{-1} \notin H$ and for all $b \in B$, either $hbh^{-1} = b$ or $hbh^{-1} \notin B$.

(ii) For all $k \in K \setminus G$, all $F \Subset K \setminus H$, there is $h \in H$ such that $k^{-1}hk \notin H$ and $hFh^{-1} \cap H = \emptyset$.

We first show that (i) implies that the normalizer is equal to LG. Take u normalizing unitary, write $u = \sum a_k u_k$, assume by contradiction that $a_{k_0} \neq 0$ for some $k \in K \setminus G$. Take a sequence $(t_n) \in \mathbb{C}K$ with norm bounded by 1 which strong-* converges to u. Note that $a_{k_0} = \tau(u u_{k_0}^*)$, so we have $\tau(t_n u_{k_0}^*)$ is far from zero for large enough n.

Consider $t_n \lambda_h t_n^*$, then this converges $*$-strongly to $u\lambda_h u^*$, all whose Fourier coefficients outside of H are zero. So the coefs of $t_n\lambda_h t_n^*$ must converge uniformly to zero outside of H. Indeed $\tau(t_n\lambda_h t_n^* u_k^*) - \tau(u\lambda_h u^* u_k^*) \leqslant \|(t_n\lambda_h t_n^* - u\lambda_h u^*)u_k^*\|_2 \leqslant \|t_n\lambda_h t_n^* - u\lambda_h u^*\|_2 \leqslant \|(t_n - u)\lambda_h u^*\|_2 + \|u\lambda_h(u - t_n)^*\|_2 \leqslant 2\,\|u - t_n\|_2$. So we need to compute $\tau(t_n\lambda_h t_n^*\lambda_k^*)$ for large enough n. Now fix such an n, write $t = t_n$, denote by (β_k) the Fourier coefficients of t. Let us see what is the Fourier coeff at $k_0 h k_0^{-1}$. We have the term $|\beta_{k_0}|^2$, but also terms coming from those k's and $j's$ in the support of t such that $khj^{-1} = k_0 h k_0^{-1}$, which is equivalent to $k_0^{-1}k = hk_0^{-1}jh^{-1}$. Now take for B the set of all $k_0^{-1}k$ for k in the support of t, and take h given by the condition, we find that this condition on k and j is met if and only if $j = k$, in particular the other terms will be squares, so the Fourier coefficient of $t\lambda_h t^*$ at $k_0 h k_0^{-1}$ is at least $|\beta_{k_0}|^2$, which is a contradiction because all those should converge uniformly to zero.

Now we show that (ii) implies (i), thus finishing the proof.

2 Basics on ergodic theory

2.1 Measure-preserving transformations and actions

Let A be the diffuse separable abelian von Neumann algebra, equipped with a trace τ. We denote by $\mathrm{Aut}(A, \tau)$ the group of trace preserving automorphisms of A. We need models for A in order to build such automorphisms.

Lemma 2.1. *Let (X, τ) be a separable topological space whose topology admits a compatible complete metric (also known as a* Polish space*). Let μ be a diffuse Borel probability measure on X. Then $\mathrm{L}^\infty(X, \mu)$ is the diffuse separable abelian von Neumann algebra.*

Proof. The difficult part is to show separability. For this, we need to show that the Hilbert space $\mathrm{L}^2(X, \mu)$ is separable. Recall that step functions are dense, so it suffices to show that the space of characteristic functions of subsets is separable. Note that the square of the L^2 distance between the characteristic function of A and B is equal to $\mu(A \triangle B)$.

The separability and metrizability ensures that the topology is second-countable, i.e. admits a countable basis (U_n). Then using the regularity of the measure (a consequence of the Polishness of the space), one sees that the set of characteristic functions of finite unions of elements of (U_n) is dense in the set of characteristic functions, so we are done. □

Definition 2.2. A **measure-preserving transformation** of a probability space (X, μ) is a bimeasurable bijection $T : X \to X$ such that $\mu(T^{-1}(B)) = \mu(B)$ for all measurable $B \subseteq X$. A **measure-preserving action** of a countable group Γ on (X, μ) is an action on (X, μ) by measure-preserving transformations.

Note that every measure-preserving transformation yields a trace preserving automorphism of $\mathrm{L}^\infty(X, \mu)$ given by $f \mapsto f \circ T^{-1}$, and that two measure-preserving transformations which yield the same automorphism are equal up to measure zero. Now every trace-preserving automorphism of a finite von Neumann algebra (M, τ) induces a unitary on its $\|\cdot\|_2$-completion $\mathrm{L}^2(M)$. In our case, the unitary associated to a measure-preserving transformation T is given by $U_T : f \mapsto f \circ T^{-1}$.

Note that we have the following formula, where m_f is the bounded operator of multiplication by f, which we will also sometimes denote by f when it is clear from the context that we work in $\mathcal{B}(\mathrm{L}^2(X, \mu))$

$$U_T m_f U_T^{-1} = m_{T \cdot f} \tag{2.1}$$

Indeed $U_T m_f U_T^{-1} g(x) = (m_f U_T^{-1} g)(T^{-1}x) = f(T^{-1}x)(U_T^{-1}g)(T^{-1}x) = f(T^{-1}x)g(x)$.

Given a measure-preserving group action of a countable group Γ, we then have an associated unitary representation $\kappa : \gamma \mapsto U_\gamma$. Moreover, the space of constant functions is invariant, we denote by $\mathrm{L}^2_0(X, \mu)$ its orthogonal. The restriction of the unitary representation κ to $\mathrm{L}^2_0(X, \mu)$, denoted by κ_0, is called the **Koopman representation** of the action.

Definition 2.3. $\Gamma \curvearrowright (X, \mu)$ is **ergodic** when $\kappa(\Gamma)' \cap \mathrm{L}^\infty(X, \mu) = \mathbb{C}1$.

Remark 2.4. Recall that $\mathrm{L}^\infty(X, \mu)$ is a masa in $\mathcal{B}(\mathrm{L}^2(X, \mu))$, i.e. $\mathrm{L}^\infty(X, \mu)' = \mathrm{L}^\infty(X, \mu)$. So equivalently, a measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is ergodic if and only if $(\pi(\Gamma) \cup \mathrm{L}^\infty(X, \mu))'' = \mathcal{B}(\mathrm{L}^2(X, \mu))$. In order to obtain a more interesting von Neumann algebra out of a measure-preserving action, we will rather consider the *crossed product construction.* The unitary representation associated to the action will also provide some relevant information.

Proposition 2.5. *The following are equivalent:*

(i) The action is ergodic;

(ii) The only elements of $\mathrm{L}^\infty(X, \mu)$ which are fixed by the action are the constant functions;

(iii) There is no measurable set A with $0 < \mu(A) < 1$ such that $\mu(\gamma A \triangle A) = 0$ for all $\gamma \in \Gamma$;

(iv) There is no measurable set A with $0 < \mu(A) < 1$ such that $\gamma A = A$ for all $\gamma \in \Gamma$;

(v) The associated Koopman representation κ_0 on $\mathrm{L}^2_0(X,\mu)$ is ergodic.

Proof. The equivalence of the first two items is a consequence of equation (2.5). The equivalence of (i) and (iii) is given by the fact that the von Neumann algebra $\kappa(\Gamma)' \cap \mathrm{L}^\infty(X,\mu)$ is generated by its projections, which are of the form χ_A, and we have $\kappa(\gamma)\chi_A\kappa(\gamma^{-1}) = \chi_{\gamma \cdot A}$. Condition (iii) clearly implies (iv), and for the converse one considers the set $\tilde{A} = \bigcap \gamma \in \Gamma\gamma A$.

Finally, it is clear that (v) implies (ii) since $\mathrm{L}^\infty(X,\mu) \subseteq \mathrm{L}^2(X,\mu)$. For the converse, we use the contrapositive. Suppose χ is a non zero invariant function in $\mathrm{L}^2_0(X,\mu)$, suppose it has norm 1. Denote by $\mathrm{L}^\infty_0(X,\mu)$ the orthocomplement of the constant functions in $\mathrm{L}^\infty(X,\mu)$. Take $f \in \mathrm{L}^\infty_0(X,\mu)$ which is 1/8-close to χ in $\|\cdot\|_2$. Then $\|f - \gamma f\|_2 \leqslant 1/4$ for all $\gamma \in \Gamma$. Since $\|f\|_2 \geqslant 7/8$, we have $\|f-\gamma f\|_2 \leqslant \frac{2}{7}\|f\|_2$. By Theorem A.4, the closed convex hull of $\pi(\Gamma)f$ contains a non zero Γ-fixed point. But recall that the operator norm ball $(\mathrm{L}^\infty_0(X,\mu))_{\|f\|}$ is closed in $\|\cdot\|_2$ and convex, so the non-zero fixed point we found actually belongs to $\mathrm{L}^\infty_0(X,\mu)$ □

2.2 Compact examples

All the transformations considered in this section are gradually more general: the odometer is a special case of profinite action, and every profinite action is a special case of a compact action.

2.2.1 The odometer

Consider the standard Borel space $X = \{0,1\}^{\mathbb{N}}$ equipped with the non-atomic probability measure $\mu = (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes\mathbb{N}}$. Any finite binary sequence $s \in \{0,1\}^n$ defines a subset N_s of the product space $\{0,1\}^{\mathbb{N}}$ consisting of all the sequences starting by s, i.e.

$$N_s := \{x \in \{0,1\}^{\mathbb{N}} : x_i = s_i \text{ for } i \in \{0,...,n-1\}\}.$$

We can see elements $a \in \{0,1\}^n$ and $b \in \{0,1\}^{\mathbb{N}} \cup \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ as words in $\{0,1\}$, and denote their concatenation by $a \frown b$. For $\epsilon \in \{0,1\}$ and $n \in \mathbb{N}$, ϵ^n is the word $(x_i)_{i=1}^n \in \{0,1\}^n$ defined by $x_i = \epsilon$.

Now the **odometer** $T_0 \in \mathrm{Aut}(X,\mu)$ is defined by $T_0((x_i)_{i\in\mathbb{N}}) = 0^{n-1}1 \frown (x_i)_{i>n}$, where n is the first integer such that $x_n = 0$, and $T_0((1,1,...,1...)) = (0,0,...,0,...)$ when such an n does not exist. This can be understood as adding $(1,0,0,...)$ to $(x_i)_{i\in\mathbb{N}}$ with right carry.

The key feature of the odometer is that his action on the sets N_s is very simple to understand as we will now see. In what follows, $\mathfrak{S}_F$ denotes the permutation group of a finite set F.

Let $n \in \mathbb{N}$, then we define a finite odometer $\sigma_n \in \mathfrak{S}_{\{0,1\}^n}$ by

$$\sigma_n((s_i)_{i=0}^{n-1}) = \begin{cases} 0^n & \text{if } (s_i) = 1^n \\ 0^{k-1}1 \frown (s_i)_{i>k} & \text{else, where } k \text{ is the first integer such that } s_k = 0. \end{cases}$$

Observe that by definition, for every $s \in \{0,1\}^n$, we have $T_0(N_s) = N_{\sigma_n(s)}$. Since the finite odometer is transitive on $\{0,1\}^n$, we conclude that the sets N_{0^n}, $T_0(N_{0^n})$,..., $T_0^{2^n-1}(N_{0^n})$ form a partition of X.

Exercise 2.1. Prove that the odometer induces a free ergodic $\mathbb{Z}$-action.

2.2.2 Profinite actions

Definition 2.6. Let Γ be a countable group. Suppose we have a decreasing sequence (Γ_n) of finite index subgroups of Γ. We then have a sequence of quotients Γ/Γ_n as well as projections $\pi_n : \Gamma/\Gamma_{n+1} \to \Gamma/\Gamma_n$. The projective limit $X = \operatorname{proj\,lim} \Gamma/\Gamma_n$ is endowed with the limit of the normalized counting measures μ_n. We then have a **profinite action** $\Gamma \curvearrowright (\operatorname{proj\,lim}_n \Gamma/\Gamma_n, \operatorname{proj\,lim}_n \mu_n)$.

2.2.3 Some compact actions

Let K be a compact Polish group equipped with its Haar measure μ. If Γ is a countable subgroup of K, the action of K by left translation on K is measure-preserving since the Haar measure is left-invariant. One can further refine this by adding in the picture a closed subgroup L and make Γ act on K/L by left translation, where K/L is equipped with the pushforward of the Haar measure. The actions obtained this way are a special case of compact actions[1], which we will define in the next section. Let us start by seeing why profinite actions are compact.

2.3 Other examples

2.3.1 Actions by automorphism on compact groups

2.3.2 Bernoulli shifts

3 Using the Koopman representation

3.1 The von Neumann ergodic theorem

3.2 Variations on ergodicity

We have seen in the previous section that the Koopman representation sees the ergodicity of the action. We will now introduce two refinements of ergodicity which are also detected by the unitary representation via some natural definition.

Definition 3.1. A measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is called **mixing** if for every Borel $A, B \subseteq X$ and every $\epsilon > 0$, for all but finitely many $\gamma \in \Gamma$ we have

$$|\mu(A \cap \gamma B) - \mu(A)\mu(B)| < \epsilon.$$

Definition 3.2. A measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is called **weakly mixing** if for every Borel $A_1, ..., A_n, B_1, ..., B_n \subseteq X$ and every $\epsilon > 0$, there is $\gamma \in \Gamma$ such that for all $i = 1, ..., n$, we have

$$|\mu(A_i \cap \gamma B_i) - \mu(A_i)\mu(B_i)| < \epsilon.$$

Observe that every mixing action is weakly mixing, and that every weakly mixing action is ergodic: if A is Γ-invariant, the condition yields $|\mu(A) - \mu(A)^2| < \epsilon$ for every $\epsilon > 0$, so $\mu(A)^2 = \mu(A)$, and we conclude $\mu(A) \in \{0, 1\}$.

[1] We will see that in the ergodic case, these are the *only* compact actions.

Proposition 3.3. *Let $\Gamma \curvearrowright (X,\mu)$ be a measure-preserving action. Then it is mixing iff the associated Koopman representation is mixing.*

Proof. Suppose that the action is mixing, recall that the mixing condition for a unitary representation needs only to be checked on a spanning subset, so here we can take elements of the form $\mu(B)\chi_A - \mu(A)\chi_B$, and a straightforward computation shows that they behave as expected.

Conversely, suppose the unitary representation is mixing, and consider what happens for the vector $\mu(B)\chi_A - \mu(A)\chi_B$... □

3.3 Compact actions

Theorem 3.4. *Let $\Gamma \curvearrowright (X,\mu)$ be a faithful ergodic compact action. Then there is a compact Polish group K and a closed subgroup $L \leqslant K$ such that Γ is a dense subgroup of K and $\Gamma \curvearrowright (X,\mu)$ is conjugate to the Γ-action by translation on K/L endowed with the pushforward of the Haar measure.*

Proof. We already know from Proposition A.12 that the closure of Γ in $\mathrm{Aut}(X,\mu)$ is a compact group K. Our first task is to lift the inclusion of K in $\mathrm{Aut}(X,\mu)$ (which is ergodic since the Γ action is ergodic) to a genuine action of K on a standard probability space. To this end, we will find a G-invariant dense separable C^*-subalgebra of L^∞ onto which K acts continuously, using the following concept from [GW05]. Call a function $f \in \mathrm{L}^\infty(X,\mu)$ K**-continuous** if the map $g \mapsto g \cdot f$ is continuous where on the right hand side we put the L^∞ norm[2].

Claim. The space of K-continuous functions is dense in $\mathrm{L}^\infty(X,\mu)$ for the L^2 norm.

Proof of the claim. Let λ be the Haar measure on G. Let (F_n) be a decreasing family of closed neighborhoods of the identity in K with trivial intersection. Let (ψ_n) be a sequence of continuous positive functions on G of integral 1 such that $\operatorname{supp}\psi_n \subseteq F_n$.

Now for $f \in \mathrm{L}^\infty(X,\mu)$ and $\psi \in \mathcal{C}(G)$, define the operator $\psi * f$ by

$$\begin{aligned}
\langle \psi * f\xi, \eta\rangle &= \int_K \psi(g)\,\langle f\kappa(g)^*\xi, \kappa(g)^*\eta\rangle\, d\lambda(g)\\
&= \int_K \psi(g)\,\langle \kappa(g) f\kappa(g)^*\xi, \eta\rangle\, d\lambda(g)\\
&= \int_K \psi(g)\,\langle g\cdot f\xi, \eta\rangle\, d\lambda(g).
\end{aligned}$$

Note that the expression in the first integral is continuous, so $\psi * f$ is well defined. It is bounded because the latter equation yields

$$\|\psi * f\| \leqslant \|\psi\|_\infty \|f\|_\infty . \tag{3.1}$$

It also belongs to $\mathrm{L}^\infty(X,\mu)$ because it commutes with the elements of $\mathrm{L}^\infty(X,\mu)$, as a straightforward computation shows[3]. Let us now compute $\|\psi * f - h\cdot\psi * f\|_\infty$. Observe

[2] Note that by assumption this map is continuous for the L^2 norm.

[3] This whole proof is actually easier if one is willing to consider vector valued integration as in [Rud07, Chap. 3]. The reader is encouraged to write down such a proof !

that $h \cdot \psi * f$ is given by

$$\begin{aligned}\langle h \cdot (\psi * f)\xi, \eta\rangle &= \langle \kappa(h)(\psi * f)\kappa(h)^*\xi, \eta\rangle \\ &= \langle \psi * f\kappa(h)^*\xi, \kappa(h)^*\eta\rangle \\ &= \int_K \psi(g)\,\langle f\kappa(hg)^*\xi, \kappa(hg)^*\eta\rangle\, d\lambda(g) \\ &= \int_K \psi(h^{-1}g)\,\langle f\kappa(g)^*\xi, \kappa(g)^*\eta\rangle\, d\lambda(g),\end{aligned}$$

so we have $h \cdot \psi * f = (h \cdot \psi) * f$ as one should expect. Since ψ is uniformly continuous and the map $\varphi \mapsto \varphi * f$ is linear, we deduce from (3.1) that $\psi * f$ is K-continuous. We finally check that $\psi_n * f \to f$ in L^2 norm. First note that by continuity and the fact that the supports of the ψ_n's have vanishing diameter, for all ϵ, for large enough n, we have $\|g \cdot f - f\|_2 \leqslant \epsilon$ for all g in the support of ψ_n. The desired conclusion follows from the estimates[4]

$$\begin{aligned}\|\psi_n * f\|_2^2 &= \int_K \psi_n(g)\,\langle g \cdot f, \psi_n(f)1\rangle \\ &= \int_K \psi_n(g) \int_K \overline{\psi}_n(h)\,\langle g \cdot f, h \cdot f\rangle \\ &\to \|f\|_2^2\end{aligned}$$

and

$$\langle \psi_n * f, f\rangle = \int_K \psi_n(g)\,\langle g \cdot f, f\rangle\, d\lambda(g) \to \|f\|_2^2,$$

which use $\int_K \psi_n = 1$. $\square_{\text{claim}}$

Now observe that K-continuous functions form a closed *-algebra in the L^∞ norm. Applying the claim, we find a countable family of functions $f_n \in \mathrm{L}^\infty$ which are K-continuous, and that generate a dense C*-algebra of $\mathrm{L}^\infty(X, \mu)$. Let A be the C*-algebra generated by these functions, which only consists of K-continuous functions. Denote by Y the spectrum of A, which is a compact Polish space acted upon continuously by K.

By construction μ induces a probability measure ν on Y, and the standard representation of A on $\mathrm{L}^2(Y, \nu)$ is isomorphic to the representation on $\mathrm{L}^2(X, \mu)$, so by taking weak closures $\mathrm{L}^\infty(X, \mu)$ is naturally isomorphic to $\mathrm{L}^\infty(Y, \nu)$.

This means that we may as well assume that we have a compact group K acting continuously on a compact space X and that Γ is a subgroup of K. Then X/K is compact, μ induces a probability measure on X/K, and the latter has to consist of one single atom, otherwise the Γ action would not be ergodic. In other words, we can also assume that the K-action is transitive. So up to isomorphism of pmp actions, Γ is acting on K/L for some closed subgroup L of K.

We are left with showing that ν is the pushforward of the Haar measure λ_K on K. Denote by λ_L the Haar measure on L. Let X_0 be a fundamental domain for the right L-action on K, so that X_0 is in bijection with K/L via the projection map, and we still denote by ν the induced probability measure on X_0.

[4] Using vector valued integration, this would follow directly from [Rud07, Thm. 3.29].

Define a probability measure on K by

$$\mu(f) = \int_{X_0} \int_L f(xl) d\lambda_L(l) d\nu(x)$$

Observe that μ is left K-invariant[5], so it has to be the Haar measure λ_K. Denote by ν' the pushforward of μ on K/L via the projection π, then for all Borel function f on K/L we have

$$\nu'(f) = \int_{X_0} \int_L f(\pi(xl)) d\lambda_L(l) d\nu(x) = \int_{X_0} f(x) d\nu(x)$$

so $\nu = \nu'$ as wanted. □

3.4 The maximal compact factor

We can use the Koopman representation in order to find a maximal compact factor in any action, where a factor simply means a Γ-invariant von Neumann subalgebra of $\mathrm{L}^\infty(X, \mu)$. The following definition is a natural companion of Definition A.7.

Definition 3.5. Suppose $\Gamma \curvearrowright (X, \mu)$ is a pmp action. A function $f \in \mathrm{L}^\infty(X, \mu)$ is **compact** if it is compact as an element of $\mathrm{L}^2(X, \mu)$.

Lemma 3.6. *The space of compact functions in* $\mathrm{L}^\infty(X, \mu)$ *is dense in the space of compact elements of* $\mathrm{L}^2(X, \mu)$*. It is moreover a von Neumann subalgebra.*

Proof. Let us first show the density statement. For any $n > 0$, consider the truncation map $\Phi_n : \mathrm{L}^2(X, \mu) \to \mathrm{L}^\infty(X, \mu)$ defined by

$$\Phi_n(\xi)(x) = \begin{cases} \xi(x) & \text{if } |\xi(x)| < n, \\ n & \text{otherwise.} \end{cases}$$

Observe that Φ_n is a contraction for the L^2 norm, and that it is Γ-equivariant. Then if $\pi(\Gamma)\xi$ is precompact, it is clear that $\pi(\Gamma)\Phi_n(\xi)$ also is, and since $\Phi_n(\xi)$ converges to ξ in L^2 norm, the conclusion follows.

The fact that the space of compact functions is a *-subalgebra follows from the continuity of the addition, of the adjoint map and of the multiplication on bounded sets for the L^2 norm. Now the *-algebra of compact functions is closed in $\mathrm{L}^\infty(X, \mu)$ for the L^2 norm by Proposition A.8, so we can apply Corollary B.2 to conclude that it is a von Neumann subalgebra. □

Definition 3.7. The von Neumann algebra of compact functions in $\mathrm{L}^\infty(X, \mu)$ is denoted by $\mathrm{L}^\infty_c(X, \mu)$.

4 The crossed product construction

4.1 The standard representation of the crossed product

$Y = X \times \Gamma$, $\mathcal{H} = \mathrm{L}^2(Y)$, make Γ act diagonally via unitaries $u_\gamma = \kappa(\gamma) \otimes \lambda(\gamma)$ and L^∞ act on the first coordinate. We have $\gamma f \gamma^{-1} \xi(x, g) = f \gamma^{-1} \xi(\gamma^{-1} x, \gamma^{-1} g) =$

[5] This uses the right invariance of λ_L, noting that $k \cdot x$ has to be right translated by an element of L in order to belong to X_0.

$f(\gamma^{-1}x)\gamma^{-1}\xi(\gamma^{-1}x,\gamma^{-1}g) = f(\gamma^{-1}x)\xi(x)$ so the same commutation relation holds as before. Observe that the algebra generated by the f's and γ's is then spanned by elements of the form au_γ which on a basic tensor $\xi \otimes \delta_g$ as

$$au_\gamma(\xi \otimes \delta_g) = a(\xi \circ \gamma^{-1} \otimes \delta_{\gamma g}) = a\xi \circ \gamma^{-1} \otimes \delta_{\gamma g}$$

In order to identify the commutant, consider the right regular representation ρ, and L^∞ acting this time via π on the γ copy of $\mathrm{L}^2(X)$ by $\gamma \cdot f$, denote by N the associated von Neumann algebra. We have $\pi(a_1)u_\gamma(\xi \otimes \delta_g) = \pi(a_1)(\xi \circ \gamma^{-1} \otimes \delta_{\gamma g}) = a_1 \circ (g^{-1}\gamma^{-1})\xi \otimes \delta_\gamma g$, while $u_\gamma \pi(a_1)(\xi \otimes \delta_g) = u_\gamma(a_1 \circ g^{-1}\xi \otimes \delta_g) = (a_1 \circ g^{-1} \circ \gamma^{-1}\xi \otimes \delta_{\gamma g})$ so π commutes with u_γ, it also commutes with the previous representation of L^∞, and by similar computations ρ commutes with L^∞. So $N \subseteq M'$.

Consider the vector $\xi = 1_X \times \delta_e$, then this vector is cyclic for crossed product and separating. To see that it is separating, we use the algebra N. Indeed, suppose that for some $x \in M$ we have $x\xi = 0$ then we compute $x(f \otimes \delta_g)$ where $f \in \mathrm{L}^\infty$ and note that the latter is equal to $\rho(g)\pi(f)(\xi)$, so by commutation this is equal to zero, so the operator x is zero. The vector ξ is thus both cyclic and separating. Let us show that it is tracial: let $\tau(x) = \langle x\xi, \xi \rangle$, then

$$\begin{aligned} au_\gamma b u_\lambda(1_X \otimes \delta_e) &= (a(\gamma \cdot b) \otimes \delta_{\gamma\lambda})) \\ bu_\lambda a u_\gamma(1_X \otimes \delta_e) &= (b(\lambda \cdot a) \otimes \delta_{\lambda\gamma}), \end{aligned}$$

so the scalar product of these two expressions with ξ is non zero if and only if $\gamma = \lambda^{-1}$, and then in this case the first is equal $\int_X a(b \circ \gamma^{-1}) = \int_X (a \circ \gamma)b = \int_X (b(a \circ \lambda^{-1})$ which is equal to the second. So we have indeed a tracial vector.

In particular M is in standard form, and we have $au_\gamma \xi = (a \otimes \delta_\gamma)$ while $(au_\gamma)^*\xi = u_{\gamma^{-1}}\overline{a}(\xi) = (\gamma^{-1} \cdot \overline{a})u_{\gamma^{-1}}\xi = (\gamma^{-1} \cdot \overline{a} \otimes \delta_{\gamma^{-1}})$.

So the canonical antiunitary J is given by $J(f \otimes \delta_g) = (g^{-1} \cdot \overline{f} \otimes \delta_{g^{-1}})$, and one can check that it intertwines our standard representation of L^∞ with π, and $\kappa \otimes \lambda$ with ρ. Since $JMJ = M'$, we conclude that $M' = N$.

Consider now the conditional expectation onto $A = \mathrm{L}^\infty(X,\mu)$. Note that in the Hilbert space this is just the projection onto the Hilbert subspace $\mathrm{L}^2(X,\mu) \otimes \delta_e$. So $u_\gamma e_A u_\gamma^*$ is the projection onto the Hilbert subspace $\mathrm{L}^2(X,\mu) \otimes \delta_\gamma$. In particular for all $x \in M$ we get

$$\hat{x} = \sum_\gamma u_\gamma e_A u_\gamma^* \hat{x} = \sum_\gamma u_\gamma e_A u_\gamma^* x(1_X \otimes \delta_e) = \sum_\gamma u_\gamma e_A u_\gamma^* x e_A(1_X \otimes \delta_e)$$

, so $\hat{x} = \sum_\gamma u_\gamma \widehat{E_A(u_\gamma^* x)}$, but $u_\gamma E_A(u_\gamma^* x) = u_\gamma E_A(u_\gamma^* x u_\gamma^* u_\gamma) = E_A(x u_\gamma^*)u_\gamma$

Free implies A masa: take x in M commuting with A, take any function $f \in A$, then $E_A(xfu_\gamma^*) = E_A(xu_\gamma^* u_\gamma f u_\gamma^*) = E_A(xu_\gamma^*(\gamma \cdot f)) = E_A(xu_\gamma^*)\gamma \cdot f$ and on the other hand $E_A(fxu_\gamma^*) = fE_A(xu_\gamma^*)$ so the Fourier coefs a_γ of x satify: $a_\gamma f = a_\gamma(\gamma \cdot f)$ for all f.) Contradiction.

4.2 Description of the normalizing algebra of $L\Gamma$

We will now use our work on unitary representations in order to get a better understanding of the normalizer of $L\Gamma$ inside the semidirect product $\mathrm{L}^\infty(X,\mu) \rtimes \Gamma$.

Theorem 4.1. *Let $\Gamma \curvearrowright (X,\mu)$, let $M = \mathrm{L}^\infty(X,\mu) \rtimes \Gamma$, then $\mathcal{N}_M(L\Gamma) \subseteq \mathrm{L}^\infty_c(X,\mu) \rtimes \Gamma$.*

Proof. We prove this by contradiction: assuming that u is a unitary not in $\mathrm{L}_c^\infty(X,\mu) \rtimes \Gamma$, we will show that u cannot normalize $L\Gamma$, using ideas similar to those of Section 1.3. First note that the conditional expectation onto $L\Gamma$ is easily describe in terms of Fourier decomposition: if $x = \sum_\gamma a_\gamma u_\gamma$ then $\mathbb{E}_{L\Gamma}(x) = \sum_\gamma \tau(a_\gamma) u_\gamma$. By the Kaplansky theorem, we find a sequence (t_n) in the *-algebra consisting of finite linear combinations of $a_\gamma u_\gamma$'s such that $t_n \to u$ in 2-norm and $\|t_n\| \leqslant 1$.

Let $y_n = \mathbb{E}_{\mathrm{L}_c^\infty(X,\mu)\rtimes\Gamma}(t_n)$ and $x_n = t_n - y_n$. Note that $\|y\|_n \leqslant 1$, and x_n is a finite linear combination of $a_\gamma u_\gamma$'s with a_γ orthogonal to $\mathrm{L}_c^\infty(X,\mu)$, and hence a_γ is in the weakly mixing part of the unitary representation κ, by a combination of Lemma 3.6 and Theorem A.17. Finally since $u \notin \mathrm{L}_c^\infty(X,\mu) \rtimes \Gamma$ and $\|u\|_2 = 1$, we must have some $\epsilon > 0$ such that for all large enough n, $\|y_n\|_2 \leqslant 1 - \epsilon$.

Now let λ be an arbitrary element of Γ. We have

$$\|\mathbb{E}_{L\Gamma}(t_n u_\lambda t_n^*) - \mathbb{E}_{L\Gamma}(u u_\lambda u^*)\|_2 \leqslant 2\|t_n - u\|_2 ,$$

and since $\mathbb{E}_{L\Gamma}(u u_\lambda u^*) = u u_\lambda u^*$ we deduce that $\|\mathbb{E}_{L\Gamma}(t_n u_\lambda t_n^*)\|_2 \to 1$, uniformly on $\lambda \in \Gamma$. We now remark that, using the bimodularity of the conditional expectation and the fact that $x_n \in (\mathrm{L}_c^\infty(X,\mu) \rtimes \Gamma)^\perp$,

$$\mathbb{E}_{\mathrm{L}_c^\infty(X,\mu)\rtimes\Gamma}(t_n u_\lambda t_n^*) = \mathbb{E}_{\mathrm{L}_c^\infty(X,\mu)\rtimes\Gamma}(x_n u_\lambda x_n^*) + y_n u_\lambda y_n^*$$

Taking further the conditional expectation onto $L\Gamma$, we get

$$\mathbb{E}_{L\Gamma}(t_n u_\lambda t_n^*) = \mathbb{E}_{L\Gamma}(x_n u_\lambda x_n^*) + \mathbb{E}_{L\Gamma}(y_n u_\lambda y_n^*)$$

By assumption for n large enough the term $\mathbb{E}_{L\Gamma}(y_n u_\lambda y_n^*)$ has L^2 norm at most $1-\epsilon$, while $\mathbb{E}_{L\Gamma}(t_n u_\lambda t_n^*) \to 1$ so for large enough n, uniformly on λ, we have $\|\mathbb{E}_{L\Gamma}(x_n u_\lambda x_n^*)\|_2 > \epsilon/2$.

Take such a large enough n, and fix it, letting $x = x_n$. We then write x as $\sum_{\gamma\in F} a_\gamma u_\gamma$, where F is a finite subset of Γ, and each a_γ belongs to $\mathrm{L}_c^\infty(X,\mu)^\perp$. For any $\lambda \in \Gamma$, we may then write $x u_\lambda x^*$ as

$$\begin{aligned} x u_\lambda x^* &= \sum_{\gamma_1,\gamma_2\in F} a_{\gamma_1} u_{\gamma_1\lambda\gamma_2^{-1}} \overline{a}_{\gamma_2} \\ &= \sum_{\gamma_1,\gamma_2\in F} a_{\gamma_1} (\gamma_1\lambda\gamma_2^{-1} \cdot \overline{a}_{\gamma_2}) u_{\gamma_1\lambda\gamma_2^{-1}}. \end{aligned}$$

We deduce that $\mathbb{E}_{L\Gamma}(x u_\lambda x^*) = \sum_{\gamma_1,\gamma_2\in F} \tau\left(a_{\gamma_1}(\gamma_1\lambda\gamma_2^{-1}) \cdot \overline{a}_{\gamma_2}\right) u_{\gamma_1\lambda\gamma_2^{-1}}$, and since the Γ-action is measure-preserving, for each $\gamma_1, \gamma_2 \in F$ we have

$$\tau\left(a_{\gamma_1}(\gamma_1\lambda\gamma_2^{-1}) \cdot \overline{a}_{\gamma_2}\right) = \tau\left((\gamma_1^{-1} \cdot a_1)\lambda \cdot (\gamma_2^{-1} \cdot \overline{a}_2)\right).$$

As explained before, all the vectors of the form $\gamma_1 \cdot a_{\gamma_1}$ or $\gamma_2^{-1} \cdot \overline{a}_2$ belong to the weakly mixing part of the unitary representation of Γ and there are only finitely many of them, so by taking a well chosen λ, we can make sure that the sum $\sum_{\gamma_1,\gamma_2\in F} \left|\tau\left(a_{\gamma_1}(\gamma_1\lambda\gamma_2^{-1}) \cdot \overline{a}_{\gamma_2}\right)\right|$ is less than $\epsilon/2$, contradicting the fact that $\|\mathbb{E}_{L\Gamma}(x u_\lambda x^*)\|_2 > \epsilon/2$. $\square$

In the case Γ is abelian, we can say much more. Indeed in this case, we have the following nice way of understanding $\mathrm{L}_c^\infty(X,\mu)$.

Proposition 4.2. *Suppose* $\Gamma \curvearrowright (X,\mu)$ *where* Γ *is abelian. Then* $\mathrm{L}_c^\infty(X,\mu)$.

5 An introduction to orbit equivalence

5.1 Dye's theorem

It is very useful to think of the odometer as a limit of partial isomorphisms in the following way. Consider first the set N_0, then T_0 sends N_0 to N_1 and we let φ_1 be the restriction of T_0 to N_0. To extend φ_1, we look closer to what happens on N_1 and find out that actually N_{10} is sent to N_{01} by T_0 and we let $\varphi_2 = \varphi_1 \cup T_{\restriction N_{10}}$. φ_2 is defined everywhere but on N_{11}. Now we add the restriction of T_0 to N_{110}, etc... In the end we get an increasing sequence (φ_n) of restrictions of T_0 such that for all n, dom $\varphi_n = X \setminus N_{1^n}$ and $\operatorname{rng} \varphi_n = X \setminus N_{0^n}$. We deduce that for almost every $x \in X$, there is n such that $T_0(x) = \varphi_n(x)$. For $n \geqslant 1$, the partial isomorphism φ_n is what we call a **ladder** of height 2^n.

A Unitary representations

A.1 The convexity trick

We start this appendix by a well-known trick which has numerous applications. It is based on the following important fact about Hilbert spaces, whose proof we recall.

Theorem A.1. *Let $\mathcal{H}$ be a Hilbert space, let $C \subseteq \mathcal{H}$ be a closed convex set. Then there is a unique $x \in C$ of minimal norm.*

Proof. Let $\alpha = \inf_{x \in C} \|x\|$, let (x_n) be a sequence of elements of C such that $\|x\|_n \to \alpha$. Given $n, m \in \mathbb{N}$, consider the point $\frac{1}{2}(x_n + x_m))$, which belongs to C. From the parallelogram identity, we have :

$$\left\|\frac{x_n + x_m}{2}\right\|^2 + \left\|\frac{x_n - x_m}{2}\right\|^2 = \frac{1}{2}\left(\|x_n\|^2 + \|x_m\|^2\right)$$

By convexity we have $\left\|\frac{x_n+x_m}{2}\right\|^2 \geqslant \alpha^2$, so

$$\alpha^2 + \left\|\frac{x_n - x_m}{2}\right\|^2 \leqslant \frac{1}{2}\left(\|x_n\|^2 + \|x_m\|^2\right) \to \alpha^2 [n, m \to +\infty],$$

so (x_n) is a Cauchy sequence. Denote by x is its limit, then $x \in C$ since C is closed, and $\|x\| = \lim \|x_n\| = \alpha$. The uniqueness follows by using the parallelogram identity once again: if $x, x' \in C$ were two minimizing point, then $(x + x')/2 \in C$ and so

$$\alpha^2 \leqslant \left\|\frac{x + x'}{2}\right\|^2 + \left\|\frac{x - x'}{2}\right\|^2 = \frac{1}{2}\left(\|x\|^2 + \|x'\|^2\right) = \alpha^2,$$

so $\|x - x'\|^2 = 0$, so $x = x'$. ☐

We may now present the *convexity trick* (convexitrick ?), which will appear in many different situations.

Corollary A.2. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Suppose that C is a $\pi(\Gamma)$-invariant convex closed subset which does not contain 0. Then the representation admits a non zero invariant vector: there is $\eta \in \mathcal{H} \setminus \{0\}$ such that $\pi(\gamma)\eta = \eta$ for all $\gamma \in \Gamma$.*

Proof. By the previous theorem, C has a unique element η of minimal norm, and η is non zero by the previous inequality. But since π is a unitary representation, for all $\gamma \in \Gamma$ the element $\pi(\gamma)\eta$ is the unique element of minimal norm of $\pi(\gamma)C = C$, so by uniqueness $\pi(\gamma)\eta = \eta$ as wanted. □

Remark A.3. A similar proof using the *circumcenter* shows that if an affine isometric action on a Hilbert space has a closed convex invariant set, then it has a fixed point.

In order to find invariant vectors, we will need to find an invariant closed convex set which does not contain zero. In the following results, the fact that this convex set does not contain zero follows from the fact that all its elements have a scalar product with a certain vector uniformly bounded away from zero, but there are some more subtle cases such as the proof of Theorem A.10.

Theorem A.4. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation, let $\epsilon < \sqrt{2}$. Suppose that ξ is a non zero vector such that for all $\gamma \in \Gamma$, $\|\pi(\gamma)\xi - \xi\| \leqslant \epsilon \|\xi\|$. Then there is a non-zero vector invariant vector $\eta \in \mathcal{H}$ which belongs to the closed convex hull of $\pi(\Gamma)\xi$.*

Remark A.5. It will be clear from the proof that one can moreover find an invariant vector η not too far from ξ, but we won't use that so we don't state this more quantitative version.

Proof of the theorem. For every $\gamma \in \Gamma$, we have

$$\|\pi(\gamma)\xi - \xi\|^2 = 2\|\xi\|^2 - 2\Re \langle \pi(\gamma)\xi, \xi\rangle .$$

We thus have $2\Re \langle \pi(\gamma)\xi, \xi\rangle \geqslant (2-\epsilon^2)\|\xi\|^2$.

Consider the convex set $C = \mathrm{conv}(\pi(\Gamma)\xi)$. By linearity we see that all the elements $\eta \in C$ actually satisfy the inequality

$$2\Re \langle \eta, \xi\rangle \geqslant (2-\epsilon^2)\|\xi\|^2 .$$

By continuity, such an inequality is also satisfied by elements of the closure $\overline{C}$ of C. The latter is a $\pi(\Gamma)$-invariant set since C is. We conclude the proof by applying the previous corollary. □

Here is another easy application of the convexitrick which a priori has nothing to do with Hilbert spaces. It is instructive to try to prove this statement directly first.

Proposition A.6. *Let Γ be a group, let X be a set, suppose that $\Gamma \curvearrowright X$ has only infinite orbits. Then for every finite subset $F \subseteq X$, there is $\gamma \in \Gamma$ such that $\gamma \cdot F \cap F = \emptyset$.*

Proof. Suppose not. Consider the real Hilbert space $\ell^2(X)$, and the associated unitary representation given by $\pi(\gamma)(f)(x) = f(\gamma^{-1}x)$. Observe that $\pi(\gamma)\chi_F = \chi_{\gamma\cdot F}$, and by hypothesis for all $\gamma \in \Gamma$ we have $|\gamma F \cap F| \geqslant 1$, so

$$\langle \pi(\gamma)\chi_F, \chi_F\rangle \geqslant 1$$

So if $C = \mathrm{conv}(\pi(\Gamma)\chi_F)$, every $\eta \in C$ satisfies $\langle \eta, \chi_F\rangle \geqslant 1$. Again the closure of C is a $\pi(\Gamma)$-invariant set all whose elements satisfy the previous inequality, in particular the closure of C does not contain the zero vector.

So our unitary representation has a non zero-invariant vector $\xi = \sum_{x\in X} a_x \delta_x$. Now observe that by invariance its coefficients a_x satisfy $a_x = a_{\gamma^{-1}\cdot x}$ for all $\gamma \in \Gamma$, and at least one of them is non zero. But since every Γ-orbit is infinite, this contradicts the summability of the squares of the coefficients. □

A.2 Compact representations

Recall that a subspace of a topological space is called precompact when its closure is compact. Recall that in a complete metric space, a subspace is precompact if and only if for every $\epsilon > 0$ it can be covered by finitely many balls of radius ϵ.

Definition A.7. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. A vector $\xi \in \mathcal{H}$ is called **compact** if $\pi(\Gamma)\xi$ is precompact for the Hilbert space norm.

By continuity of the sum, the space of compact vectors is a vector subspace. Let us show that it is moreover closed. Suppose $\pi(\Gamma)\xi$ is not precompact, then it contains an infinite ϵ-discrete set for some $\epsilon > 0$, which is obtained as $\{\pi(\gamma_i)\xi : i \in I\}$. Then if η is $\epsilon/3$ close to ξ, the set $\{\pi(\gamma_i)\eta : i \in I\}$ is $\epsilon/3$ discrete, so $\pi(\Gamma)\eta$ is not precompact either. We have shown:

Proposition A.8. *The space of compact vectors is a closed subspace, thus defining a subrepresentation of the representation.*

Definition A.9. A unitary representation is compact when all its vectors are compact.

We will show every compact representation splits as an infinite direct sum of finite-dimensional representations. For this, we will use compact groups and the fact that their continuous unitary representations do have this property, which is part of the Peter-Weyl theorem.

Theorem A.10. *Let G be a compact group, let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation. Then π contains a finite dimensional subrepresentation.*

Proof. Let p be any non trivial finite dimensional projection in $\mathcal{H}$, then p is a trace class operator of trace $\operatorname{Tr} p = \dim p\mathcal{H} > 0$. Denote by $\mathcal{B}_1(\mathcal{H})$ the Banach space of trace class operators. Then G acts continuously on $\mathcal{B}_1(\mathcal{H})$ by conjugacy ($\gamma \cdot x = \pi(\gamma)x\pi(\gamma)^*$). Then $G \cdot p$ is a compact subset of $\mathcal{B}_1(\mathcal{H})$ all whose elements have trace $\operatorname{Tr} p$. The closed convex hull (in $\mathcal{B}_1(\mathcal{H})$!) C of $G \cdot p$ is also compact (every closed convex hull of a compact set is compact in a Banach space), and all its elements have trace $\operatorname{Tr} p$ by continuity of Tr on $\mathcal{B}_1(\mathcal{H})$, in particular C does not contain 0.

Now the embedding of $\mathcal{B}_1(\mathcal{H})$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ of Hilbert-Schmidt operators is continuous, so by compactness C is still a compact convex subset of $\mathcal{B}_2(\mathcal{H})$, and it does contain a non zero G-fixed point x by Corollary A.2.

So x is a non-trivial compact operator belonging to the commutant of our representation π. Then it has a non-trivial finite-dimensional spectral projection which belongs to the commutant of our representation π, and the range of this projection is the desired finite dimensional subrepresentation. □

Before going back to countable groups, we need a better understanding of the precompact subgroups of the unitary group. Here by **precompact** we mean *whose closure is compact*, so this is a *relative* notion. There is a more satisfying intrinsic definition using *uniform structures*, but it would take us too far afield[6].

The unitary group is a Polish group when endowed with the strong topology, which is the topology induced by the product topology on $\mathcal{H}^{\mathcal{H}}$. But it is not closed in there,

[6] Let us just mention that the next proposition also follows the fact that a topological group is precompact for the left uniformity iff it is precompact for the right uniformity iff it is precompact for the Raikov uniformity, and that the unitary group is Raikov complete...

for instance if we fix an orthonormal basis (ξ_n), the sequence of unitaries which permutes cyclically the first n vectors converges to the isometry $\xi_n \mapsto \xi_{n+1}$.

Nevertheless, first note that $\mathcal{B}(\mathcal{H})_1$ is closed in $\mathcal{H}^{\mathcal{H}}$. The graph of the weakly continuous map $x \in \mathcal{B}(\mathcal{H})_1 \mapsto x^*$ is closed in the product of the weak topology, hence also in the strong product topology. In other words, the set of all (x, x^*) is closed in $\mathcal{B}(\mathcal{H})_1 \times \mathcal{B}(\mathcal{H})_1 \subseteq \mathcal{H}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{H}}$. Now recall that the composition is continuous for the strong toplogy when restricted to $\mathcal{B}(\mathcal{H})_1$, in particular the set of all $(x, y) \in \mathcal{B}(\mathcal{H})_1$ such that $xy = yx = 1$ is closed. Putting this together with the previous statement, we have shown:

Lemma A.11. *The map $\Phi : u \in \mathcal{U}(\mathcal{H}) \mapsto (u, u^*) \in \mathcal{H}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{H}}$ is continuous with closed image, where we equip $\mathcal{U}(\mathcal{H})$ with the strong topology, and $\mathcal{H}^{\mathcal{H}}$ with the product topology, $\mathcal{H}$ being equipped with its Hilbert norm.*

Proposition A.12. *For a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, the following are equivalent:*

(i) π is compact;

(ii) $\pi(\Gamma)$ is precompact in $\mathcal{H}^{\mathcal{H}}$ equipped with the product topology;

(iii) $\pi(\Gamma)$ is precompact in $(\mathcal{B}(\mathcal{H}))_1$ equipped with the strong operator topology;

(iv) $\pi(\Gamma)$ is precompact in $\mathcal{U}(\mathcal{H})$ equipped with the strong operator topology.

Proof. The implication (i)⇒(ii) follows from Tychonov's theorem, which ensures us that the infinite product $\prod_{\xi \in \mathcal{H}} \overline{\pi(\Gamma)\xi}$ is compact.

The equivalence between (ii) and (iii) is a direct consequence of the fact that $(\mathcal{B}(\mathcal{H}))_1$ is closed in $\mathcal{H}^{\mathcal{H}}$.

The implication (iii)⇒(iv) follows from the previous lemma, noting that $\Phi(\pi(\Gamma)) \subseteq \overline{\pi(\Gamma)} \times \overline{\pi(\Gamma)} \cap \mathcal{U}(\mathcal{H})$, where here we take the closure in $\mathcal{B}(\mathcal{H})_1$.

Finally the implication (iv)⇒(i) follows from the fact that a continuous image of a compact set is compact. □

We finally have the following characterization of compact representations.

Theorem A.13. *A unitary representation is compact if and only if it splits as a direct sum of finite dimensional representations.*

Proof. This is a direct application of the previous results along with Zorn's lemma. □

A.3 Weakly mixing representations

We now introduce a property which is orthogonal to compactness.

Definition A.14. A unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is called **weakly mixing** if given a finite set of unit vectors F, and $\epsilon > 0$, one can find an element $\gamma \in \Gamma$ such that for all $\xi, \eta \in F$,

$$|\langle \pi(\gamma)\xi, \eta \rangle| < \epsilon$$

In other words, there is a sequence of group elements (γ_n) such that the associated sequence of unitary operators $\pi(\gamma_n)$ converges weakly to 0. The adjective "weakly" is justified by the following definition.

Definition A.15. A unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is called **mixing** if given $\epsilon > 0$ and vector ξ, η, for all but finitely many elements $\gamma \in \Gamma$ we have

$$|\langle \pi(\gamma)\xi, \eta\rangle| < \epsilon$$

This is sometimes written as $\lim_{\gamma\to\infty} \langle \pi(\gamma)\xi, \eta\rangle = 0$.

Exercise A.1. Check that every mixing unitary representation is weakly mixing, and that the regular representation of any discrete group is mixing.

Using the polarization identity, one can check that we can take $\xi = \eta$ in both definitions. Let us now relate weak mixing to compactness via finite dimensional subrepresentations.

Proposition A.16. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation, then the following are equivalent:*

1. *π has a finite dimensional representation*
2. *$\pi \otimes \overline{\pi}$ has an invariant vector*

Proof. Suppose $\mathcal{K}$ is a non trivial finite dimensional π-invariant subspace, consider the projection onto $\mathcal{K}$, which is an element of $\mathcal{H}\otimes\overline{\mathcal{H}}$ (if (ξ_i) is an orthonormal basis of $\mathcal{K}$, this is the vector $\xi_i \otimes \overline{\xi}_i$). It is an invariant vector for the representation $\pi \otimes \overline{\pi}$, which acts on this projection as a conjugacy.

Conversely, suppose $\pi \otimes \overline{\pi}$ has an invariant vector, then this invariant vector is a Hilbert-Schmidt operator whose spectral projections must commute with π, so that π has a finite dimensional representation. □

Let us also note the following computation on $\pi \otimes \overline{\pi}$: suppose $(\xi_i)_{i=1}^n$ is a finite family of vectors, and let us compute the scalar product

$$\begin{aligned}\left\langle \sum_{i=1}^n \pi\otimes\overline{\pi}(\xi_i\otimes\xi_i), \sum_{j=1}^n \xi_j\otimes\xi_j \right\rangle &= \sum_{i,j=1}^n \langle \pi(\gamma)\xi_i, \xi_j\rangle\, \overline{\langle \pi(\gamma)\xi_i, \xi_j\rangle} \\ &= \sum_{i,j=1}^n |\langle \pi(\gamma)\xi_i, \xi_j\rangle|^2 .\end{aligned}$$

So we see that if π is weakly mixing, the right hand quantity can be made arbitrarily small, in particular π has not finite dimensional representation.

The following result is fundamental.

Theorem A.17. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Denote by $\mathcal{H}_c$ the space of compact vectors. Then the restriction of π to $\mathcal{H}_c^\perp$ is weakly mixing.*

Proof. By Thm. A.13, the restriction of π to $\mathcal{H}_c^\perp$ contains no finite dimensional subrepresentation.

Suppose by contradiction that π is not weakly mixing, then by the previous computation we find a $\pi\otimes\overline{\pi}$ invariant vector using convexity. This yields in turn a Hilbert-Schmidt operator in the commutant of π, and hence once of its spectral projection yields a finite dimensional subspace, a contradiction. □

B General facts about von Neumann algebras

B.1 Kaplansky's theorem

Theorem B.1. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a *-subalgebra. Let $x \in A''$. Then there is a sequence (x_n) of elements of $(A)_{\|x\|}$ such that $x_n \to x$ *-strongly.*

Corollary B.2. *Let (M, τ) be a finite von Neumann algebra in standard form, suppose N is a *-subalgebra which is closed in M under the L^2 norm. Then N is a von Neumann subalgebra of M.*

Proof. By Kaplansky's theorem, every element of the weak closure of N is obtained as the strong limit of a bounded sequence of elements of N, which by boundedness must also converge in the L^2 norm. So its limit belongs to N by assumption. □

B.2 Conditional expectations

From uniqueness $E_{\varphi(A)}(x) = \varphi(E_A(\varphi^{-1}(x)))$, applying this to inner aut we get $E_{uAu^*}(x) = uE_A(u^*xu)u^*$, so $E_{uAu^*}(uxu^*) = uE_A(x)u^*$.

C Topological groups

C.1 General definition and uniform structures

C.2 Compact groups and the Haar measure

We provide a proof that every metrizable compact group has a unique Haar measure, a result which is much easier to prove than in the more general locally compact case. In the course of the proof, we freely use the Riesz correspondence between probability measures and linear functionals on continuous functions.

Proposition C.1. *Every compact metrizable group admits a unique Borel left-invariant probability measure which is also right-invariant.*

Proof. The trick is to use the convolution of probability measures on our compact group G. Given two Borel probability measures μ and ν on G, their convolution is defined as the pushforward of the product measure $\mu \otimes \lambda$ under the group operation map. So

$$\mu * \nu(A) = \int_G \int_G \chi_A(gh) d\mu(g) d\nu(h) = \int_G \mu(Ah^{-1}) d\nu(h) = \int_G \nu(g^{-1}A) d\mu(g)$$

From the associativity of the group law, we get the associativity of the convolution. We now have the following important claim:

Claim. Let α be a Borel probability measure on G with full support. Let λ such that $\alpha * \lambda = \lambda$. Then λ is left-invariant.

Proof of the claim. Let u be a continuous function on G, define a function $\Phi_\lambda(u)$ by

$$\Phi_\lambda(u)(g) = (\delta_g * \lambda)(f) = \int_G u(gh) d\lambda(h)$$

Note that $\Phi_\lambda(u)$ is continuous as a consequence of the uniform continuity of u. For every $a \in G$, we then have

$$\begin{aligned}\int_G \Phi_\lambda(u)(ag)d\alpha(g) &= \int_G\int_G u(agh)d\alpha(g)d\lambda(h)\\ &= \delta_a * \alpha * \lambda(f)\\ &= \delta_a * \lambda(f)\\ &= \Phi_\lambda(u)(a)\end{aligned}$$

Pick $a \in G$ where the continuous function Φ_λ reaches its maximum, then since α is fully supported, this can only happen if $\Phi_\lambda(u)$ is constant. We thus have $\delta_g * \lambda = \lambda$ for all $g \in G$: the measure λ is left-invariant. $\square_{\text{claim}}$

Now note that left convolution is continuous for the weak topology: if $\lambda_n \to \lambda$ weakly and μ is fixed, and f is a continuous function, we have

$$(\mu * \lambda_n)(f) = \int_G \left(\int_G f(gh)d\mu(g)\right) d\lambda_n(h),$$

and the term in parenthesis is again continuous as a function of h because f is uniformly continuous, so we do have $\mu * \lambda_n \to \mu * \lambda$ weakly.

Now let α be any full supported measure on G (for instance $\alpha = \sum_{n\geqslant 1} 2^{-n}\delta_{g_n}$, where (g_n) is dense in G)). Let λ be a weak limit point of the sequence

$$\lambda_n := \frac{\alpha + \alpha * \alpha + \cdots + \alpha^{*n}}{n}$$

For every n we have $\|\alpha * \lambda_n - \lambda_n\| \leqslant \frac{2}{n}$, in particular $\alpha * \lambda_n - \lambda_n$ converges weakly to zero, so by continuity we have $\alpha * \lambda = \lambda$. By the previous claim, we have just built the desired left-invariant probability measure.

Observe that λ has full support, because its support is G-invariant. Now note that the claim also yields that if $\lambda * \alpha = \lambda$ for some fully supported α, then λ is right-invariant. We have $\delta_g * \lambda = \lambda$ for every $g \in G$, so by integrating, $\lambda * \lambda = \lambda$, so applying the right-invariant version of the claim to $\alpha = \lambda$, we conclude that λ is G invariant.

Finally if μ were another such left-invariant probability measure, by the same arguments it would have full support and be right invariant, but then $\lambda * \mu = \lambda = \mu$. $\square$

C.3 Polish groups associated to von Neumann algebras

Given a finite von Neumann algebra (M, τ), every trace preserving automorphism of M induces a unitary on $\mathrm{L}^2(M, \tau)$, which allows one to see $\mathrm{Aut}(M, \tau)$ as a subgroup of the unitary group of $\mathrm{L}^2(M, \tau)$.

Proposition C.2. $\mathrm{Aut}(M, \tau)$ *is a closed subgroup of* $\mathcal{U}(\mathrm{L}^2(M, \tau))$.

Proof. Let $\alpha \in \mathrm{Aut}(M, \tau)$. The fact that α commutes with $*$ is witnessed by the fact that the associated unitary operator commutes with J. The fact that α preserves the unit ball of M defines a closed condition because the latter is a closed subset of $\mathrm{L}^2(M, \tau)$. Finally the fact that α commutes with multiplication on the unit ball of M defines a closed condition by continuity.

In other words, $\mathrm{Aut}(M, \tau)$ is the set of unitaries which:

- Commute with J
- Leave the unit ball of M invariant
- Commute with multiplication on the unit ball of M,

and so it is closed. □

Denote by κ the injection of $\mathrm{Aut}(M,\tau)$ into $\mathcal{U}(\mathrm{L}^2(M,\tau))$. We can now observe the following conjugation relation for all $\alpha \in \mathrm{Aut}(M,\tau)$:

$$\alpha(x) = \kappa(\alpha)x\kappa(\alpha)^*$$

Indeed, it suffices to show that $\alpha(x)\kappa(\alpha)\hat{y} = \kappa(\alpha)x\hat{y}$ for all $y \in M$, which is true since $\alpha(x)\kappa(\alpha)\hat{y} = \alpha(x)\widehat{\alpha(y)} = \widehat{\alpha(x)\alpha(y)} = \widehat{\alpha(xy)}$ and $\kappa(\alpha)\widehat{xy} = \widehat{\alpha(xy)}$.

References

[GW05] E. Glasner and B. Weiss. Spatial and non-spatial actions of Polish groups. *Ergodic Theory and Dynamical Systems*, 25(05):1521, August 2005.

[Rud07] Walter Rudin. *Functional Analysis*. Mcgraw Hill Higher Education, New Delhi, 2007.