Contents

Chapter 1

Topological spaces and metrics

One of the main purposes of descriptive set theory is to study the complexity of sets, or rather subsets of *Polish spaces*, which will be defined in the next chapter after the following thorough review of some fundamental notions from topology.

We chose to present topological spaces first in terms of neighborhood systems which provide a more direct grasp on the notion of continuity. The usual definition in terms of a family of *open* sets satisfying certain axioms is then introduced. Open sets are fundamental in descriptive set theory since we them as the simplest subsets of a topological space: if we know that x belongs to an open sets, we know that all the points sufficiently close to it also do.

1.1 Topological spaces

1.1.1 A motivating example: metric spaces

Let us start by recalling that a **metric space** is a set X equipped with a **metric**, i.e. a map $d : X \times X \to [0, +\infty[$ which satisfies the following three axioms:

- (separation) for every $x \in X$, $d(x, x) = 0$ and for every $x, y \in X$, if $x \neq y$ then $d(x, y) > 0$;
- (symmetry) for every $x, y \in X$, one has $d(x, y) = d(y, x)$;
- (triangle inequality) for every $x, y, z \in X$, one has the inequality

$$d(x, z) \leqslant d(x, y) + d(y, z).$$

The first example of a metric space that one encounters is the reals equipped with the distance $d(x, y) = |x - y|$. More generally normed vector spaces $(V, \|\cdot\|)$ are naturally endowed with a metric $d(x, y) = \|x - y\|$, so other examples of metric spaces include Hilbert spaces, L^p spaces etc.

The reader has probably already encountered the following important consequences of the triangle inequality, so we only state them and leave their proofs in the following exercise.[1].

Exercise 1.1. Let (X, d) be a metric space.

[1] Remember that all the exercises of this book have indications and solutions to be found on .

1. Show that for every $x, y, z \in X$ we have
$$d(x,z) \geqslant |d(y,x) - d(y,z)| \tag{1.1}$$

2. Let A be a non-empty subset of X, let $x \in X$ and define $d(x,A) := \inf_{a\in A} d(x,a)$. We say that the map $x \mapsto d(x,A)$ is the distance to A. Show that for all $x, y \in X$ we have
$$d(x,y) \geqslant |d(x,A) - d(y,A)|\,. \tag{1.2}$$
How is this a generalization of the previous question ?

One of the goals of these lectures is to present a wealth of important examples of metric spaces. Contrarily to the normed vector space case, these spaces will often not come with a canonical metric, and it will be better to view them as topological spaces. But before we get into this, we will recall some important definitions in the metric setup so as to make the definition of a topological space more natural.

Definition 1.1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Given $x_0 \in X$ and $y_0 \in Y$, one says that $f(x)$ **tends to** y_0 as x tends to x_0 if for every $\epsilon > 0$, there is $\delta > 0$ such that whenever $x \in X$ verifies $d_X(x, x_0) < \delta$, its image $f(x)$ satisfies to $d_Y(y_0, f(x)) < \epsilon$.

By the separation axiom, if $f(x)$ tends to y_0 as x tends to x_0 then one has $f(x_0) = y_0$. So the y_0 in the above definition is unique, and when $f(x)$ tends to y_0 as x tends to x_0 we will write
$$\lim_{x\to x_0} f(x) = y_0.$$
This can be rephrased in terms of open balls. Given $x \in X$, and $r > 0$, the **open ball** $B_d(x,r)$ **of radius** r **around** x consists in all the points which are r-close to x:
$$B_d(x,r) = \{y \in X : d(x,y) < r\}.$$
When the metric d is clear from the context, we will skip the index and simply write $B(x,r)$ for the open ball of radius r around x.

Given a subset $B \subseteq X$, we say that B is an open ball around $x \in X$ if it is equal to $B_d(x,r)$ for some $r > 0$.

We now see that the definition of convergence is equivalent to the following: given any open ball B_1 around y_0, one can find an open ball B_2 around x whose image $f(B_2)$ is contained in B_1.

A slightly more efficient to say this is: the preimage of any open ball around $f(x_0)$ contains an open ball around x_0. Who can do more can do less, so we may as well say that the preimage of any set containing a ball around $f(x_0)$ contains a ball around x_0.

We have thus obtained a symmetric definition of convergence purely in terms of preimages, which is nice since as the reader knows preimages are well-behaved with respect to set-theoretic operations. The notion of *neighborhood* captures what is going on here.

Definition 1.2. Let (X, d) be a metric space. Given $x \in X$, a subset $V \subseteq X$ is a **neighborhood** of x if it contains an open ball around x.

As explained earlier, we have $\lim_{x\to x_0} f(x) = y_0$ if and only if the preimage $f^{-1}(V)$ of any neighborhood V of y_0 is a neighborhood of x_0.

1.1.2 Neighborhood systems

Definition 1.3. Let (X, d) be a metric space. Given $x \in X$, we define $\mathcal{V}_x$ the **neighborhood filter** of x to be the set of neighborhoods of x:

$$\mathcal{V}_x := \{V \subseteq X : \exists r > 0, B(x,r) \subseteq V\}.$$

Proposition 1.4. *The neighborhood filter $\mathcal{V}_x$ of a point x in a metric space (X, d) satisfies the following properties.*

(i) It contains X: $X \in \mathcal{V}_x$.

(ii) It is stable under finite intersection[2] : for every $V, W \in \mathcal{V}_x$ we have $V \cap W \in \mathcal{V}_x$.

(iii) It is stable under taking supersets: for all $V \in \mathcal{V}_x$, if $W \subseteq X$ satisfies $V \subseteq W$ then $W \in \mathcal{V}_x$.

(iv) all its elements contain x: for all $V \in \mathcal{V}_x$, we have $x \in V$.

Proof. (i) holds because $B(x, 1) \subseteq X$. (ii) follows from the fact that $B(x,r) \cap B(x,r') = B(x, \min(r,r'))$. (iii) is straightforward from the definition and (iv) is a consequence of the fact that every open ball around x contains x. □

Remark 1.5. In general a **filter** on X is a set of subsets of X containing X, stable under finite intersections and supersets, and which does not contain the empty set. Note that neighborhood filters are examples of filters. Filters provide a very useful way of defining convergence when dealing with non-metric spaces. The interested reader is referred to Exercise **??** for a proof of the Tychonov theorem using filters.

We will now give an important property of the neighborhood filters in a metric space by seeing how the triangle inequality relates them.

Lemma 1.6. *Let (X, d) be a metric space and $x \in X$. Then the open ball $B(x,r)$ is a neighborhood of all its points: for all $y \in B(x,r)$, one has $B(x,r) \in \mathcal{V}_y$.*

Proof. Let $y \in B(x,r)$. Then for all $z \in B(y, r - d(x,y))$, the triangle inequality yields

$$\begin{aligned} d(x,z) &\leqslant d(x,y) + d(y,z) \\ &< d(x,y) + r - d(x,y) = r. \end{aligned}$$

We thus have the inclusion $B(y, r - d(x,y)) \subseteq B(x,r)$. We conclude that the open ball $B(y, r - d(x,y))$ witnesses the fact that $B(x,r)$ is a neighborhood of y. □

Proposition 1.7. *Let (X, d) be a metric space. The family of neighborhood filters $(\mathcal{V}_x)_{x \in X}$ of a point x in a metric space (X, d) satisfies the following property.*

(v) For every $x \in X$ and every neighborhood $V \in \mathcal{V}_x$, there exists a neighbourhood $W \in \mathcal{V}_x$ such that V is a neighbourhood of every $y \in W$ ($\forall y \in W, V \in \mathcal{V}_x$).

Proof. Let $x \in X$, let V be a neighbourhood of x. By definition there is $r > 0$ such that $B(x,r) \subseteq V$. Let $W = B(x,r)$, then Lemma 1.6 implies that $W \in \mathcal{V}_y$ for all $y \in W$. Since $W \subseteq V$ and every $\mathcal{V}_y$ is a filter, we conclude that for all $y \in W$ we have $V \in \mathcal{V}_y$. □

[2] An immediate recurrence shows that $\mathcal{V}_x$ is stable under finite intersection if and only if the intersection of any *two* of its elements still belongs to it.

The above property is fundamental since it is the only one which relates neighborhoods from different points. It is moreover the only one for which we used the triangle inequality.

We will now turn these propositions into definitions from which we will recover the commonly used definition of a topology.

Definition 1.8. Let X be a set. A **neighborhood system** on X is a family $(\mathcal{V}_x)_{x\in X}$ of subsets of X satisfying the following conditions for all $x \in X$.

(i) $X \in \mathcal{V}_x$.

(ii) For all $V, W \in \mathcal{V}_x$ we have $V \cap W \in \mathcal{V}_x$.

(iii) For all $V \in \mathcal{V}_x$ if $W \subseteq X$ satisfies $V \subseteq W$ then $W \in \mathcal{V}_x$.

(iv) For all $V \in \mathcal{V}_x$ we have $x \in V$.

(v) For all $V \in \mathcal{V}_x$ there is $W \in \mathcal{V}_x$ such that for all $y \in W$ we have $V \in \mathcal{V}_x$.

The conjunction of Proposition 1.4 and Proposition 1.7 can now be rephrased as: the family of neighborhood filters in a metric space is a neighborhood system. Let us give another example.

Definition 1.9. The **one-point compactification of** $\mathbb{N}$ is the set $\overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ equipped with the following neighborhood system :

- For every $n \in \mathbb{N}$, every subset of $\overline{\mathbb{N}}$ containing n is a neighborhood of n.
- The neighborhoods of $+\infty$ are subsets of $\overline{\mathbb{N}}$ containing an interval of the form

$$[[n, +\infty]] := \{n \in \mathbb{N} : n \geqslant N\} \cup \{+\infty\}.$$

Exercise 1.2. Check that this is indeed a neighborhood system and then find a metric on $\overline{\mathbb{N}}$ which induces the same neighborhood system.

1.1.3 Topologies

Definition 1.10. Let $\mathcal{V} = (\mathcal{V}_x)_{x\in X}$ be a neighborhood system on a set X. A subset U of X is $\mathcal{V}$**-open** if it is a neighborhood of all its elements: for all $x \in U$ one has $U \in \mathcal{V}_x$.

Example 1.11. If (X, d) is a metric space, then its open balls are open by Lem. 1.6.

The properties satisfied by the family of $\mathcal{V}$-open sets of a neighborhood system $\mathcal{V}$ motivate the following definition.

Definition 1.12. Let X be a set. A **topology** on X is a set τ of subsets of X called *open sets* such that:

(a) the empty set and the whole set X are open: $\emptyset, X \in \tau$;

(b) the intersection of any *finite* family of open sets is open: for all $U, V \in \tau$ we have $U \cap V \in \tau$;

(c) the reunion of any family of open sets is open: for all $\mathcal{U} \subseteq \tau$ we have $\bigcup_{U\in\mathcal{U}} U \in \tau$.

A couple (X, τ) where τ is a topology on X is called a **topological space**.

Proposition 1.13. *The $\mathcal{V}$-open sets of a neighborhood system $\mathcal{V}$ form a topology on X.*

Proof. The empty set is a neighborhood of all its elements, hence it is open. Also X is a neighborhood of all its points by property (i), so property (a) is established.

If U, U' are open and $x \in U \cap U'$, we find $V, V' \in \mathcal{V}_x$ such that $V \subseteq U$ and $V' \subseteq U'$. So $V \cap V' \subseteq U \cap U'$ and property (ii) implies $V \cap V' \in \mathcal{V}_x$ which proves property (b).

Finally if $\mathcal{U}$ is a family of open sets and $x \in \bigcup \mathcal{U}$, there is $U \in \mathcal{U}$ such that $x \in U$ and since U is open we find a neighborhood $V \in \mathcal{V}_x$ such that $V \subseteq U \subseteq \bigcup \mathcal{U}$. So $\bigcup \mathcal{U}$ is open which proves property (c). □

Definition 1.14. Let $\mathcal{V}$ be a neighborhood system, then the set of $\mathcal{V}$-open sets is called the **topology associated to the neighborhood system $\mathcal{V}$**.

Example 1.15. If X is a set, then $\mathcal{P}(X)$ is clearly a topology. It is called the **discrete topology**. Observe that it can also be viewed as the topology associated to the neighborhood system $(\mathcal{V}_x)_{x \in X}$ where for each $x \in X$ the neighborhood filter $\mathcal{V}_x$ is the set of all $V \subseteq X$ such that $x \in V$.

Let us remark that only the two first properties (i) and (ii) of neighborhood systems have been used in the proof that neighborhood systems define a topology. The three other ones will be used to go the other way around and recover our initial neighborhood system from the associated topology.

Definition 1.16. Let τ be a topology. For $x \in X$ let $\mathcal{V}_x^\tau$ be the set of $V \subseteq X$ containing an open set containing x. Then $(\mathcal{V}_x^\tau)_{x \in X}$ is the **neighborhood system associated to the topology τ**.

Let us check that $(\mathcal{V}_x^\tau)_{x \in X}$ is indeed a neighborhood system.

Proof. Let $x \in X$. Since X is open it is a neighborhood of all its points (property (a)) so $X \in \mathcal{V}_x^\tau$: property (i) holds.

Let $V_1, V_2 \in \mathcal{V}_x$ and let $U_1, U_2 \in \tau$ be open sets containing x such that $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Then by property (b) the intersection $U_1 \cap U_2$ is open, and since $U_1 \cap U_2 \subseteq V_1 \cap V_2$ we conclude that $V_1 \cap V_2 \in \mathcal{V}_x^\tau$, establishing property (ii).

Property (iii) follows from the fact that if V contains an open set containing x then all the sets containing V also do. Property (iv) also follows directly from the definition of $(\mathcal{V}_x^\tau)_{x \in X}$.

Finally if $V \in \mathcal{V}_x^\tau$, let $U \subseteq V$ be open containing x. Then $U \in \mathcal{V}_x^\tau$ and since U is open we have $V \in \mathcal{V}_y$ for all $y \in U$. This proves property (v). □

Now that we have established a correspondence between topologies and neighborhood systems, we need to check that this correspondence is bijective, i.e. that the "maps" provided by definitions 1.14 and 1.16 are inverse of each other.

Theorem 1.17. *Let X be a set, let τ be a topology on X and let $(\mathcal{V}_x)$ be a neighborhood system on X.*

(1) The topology associated to the neighborhood system associated to τ is equal to τ.

(2) The neighborhood system associated to the topology associated to $(\mathcal{V}_x)_{x \in X}$ is equal to $(\mathcal{V}_x)_{x \in X}$.

Proof. (1) Let τ' be the topology associated to the neighborhood system $(\mathcal{V}_x^\tau)_{x\in X}$ associated to τ. By definition an open set is a neighborhood of all its points so $\tau \subseteq \tau'$.

Conversely, let $U \in \tau'$ and let $x \in U$. By definition of τ' we find $V \in \mathcal{V}_x^\tau$ such that $x \in V \subseteq U$. By the definition of $\mathcal{V}_x^\tau$ we then find a τ-open set O containing x such that $O \subseteq U$. We conclude that U is the union of the τ-open sets contained in U, so since τ is stable under arbitrary unions we conclude that $U \in \tau$. Hence we have the reverse inclusion $\tau' \subseteq \tau$, establishing (1).

(2) Let $\tau_{\mathcal{V}}$ be the topology associated to $(\mathcal{V}_x)_{x\in X}$, and let $(\mathcal{V}'_x)_{x\in X}$ be the associated neighborhood system. For all $x \in X$, if $V \in \mathcal{V}'_x$, we have a $\tau_{\mathcal{V}}$-open set U containing x such that $U \subseteq V$. By the definition of $\tau_{\mathcal{V}}$ we have that $U \in \mathcal{V}_x$, and since $U \subseteq V$ and $\mathcal{V}_x$ is stable under taking supersets, we conclude that $V \in \mathcal{V}_x$. Hence $\mathcal{V}'_x \subseteq \mathcal{V}_x$.

The reverse inclusion is where condition (v) on neighborhood systems comes into play (recall that this condition reflects the triangle inequality in metric spaces). Let $x \in X$ and let $V \in \mathcal{V}_x$. Now consider the set $U = \{y \in X : V \in \mathcal{V}_y\}$. By property (iv) the set U is contained in V and contains x. Now property (v) implies that U is $\tau_{\mathcal{V}}$-open: if $y \in U$ then $V \in \mathcal{V}_y$ so we find $W \in \mathcal{V}_y$ such that $W \subseteq V$ and $V \in \mathcal{V}_z$ for all $z \in W$. So $W \subseteq U$ and we conclude that U is indeed $\tau_{\mathcal{V}}$-open. We conclude that $V \in \mathcal{V}'_x$, so $\mathcal{V}'_x = \mathcal{V}_x$ as wanted. □

The above proof made implicitly use of an important concept in a topological space (X, τ): the **interior** $\mathring{A}$ of a subset $A \subseteq X$, defined to be the reunion of the open sets contained in A:

$$\mathring{A} = \bigcup \{U \in \tau : U \subseteq A\}.$$

Since a topology is stable under arbitrary unions the interior of A is actually an open set, and by construction it is the greatest open set contained in A. In particular A is open if and only if $A = \mathring{A}$.

We can now reformulate some concepts from the proof of Theorem 1.17: in the proof that $\tau' \subseteq \tau$ in (1), we showed that if $U \in \tau'$ then U is equal to its τ-interior, hence τ-open. Moreover the essence of the proof that $\mathcal{V} \subseteq \mathcal{V}'$ in (2) is the following statement which we leave as an exercise.

Exercise 1.3. Let $\mathcal{V}$ be a neighborhood system on a set X. Show that given a subset $A \subseteq X$, the $\tau_{\mathcal{V}}$-interior of A is the set of all $x \in X$ such that A is a neighborhood of x.

If (X, d) is a metric space, we can associate to it a neighborhood system as in Definition 1.3 and thus a topology via Definition 1.14. This topology will be called the **topology associated to the metric** d. By definition, a subset U of X is open for the topology associated to d if for every $x \in U$ there is $\epsilon > 0$ such that $B_d(x, r) \subseteq U$. Note that every open ball is indeed open for this topology as a consequence of Lemma 1.6.

Example 1.18. Let X be a set equipped with the discrete topology. Then the **discrete metric** δ defined by

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

is compatible with the discrete topology.

Definition 1.19. Given a topological space (X, τ), a metric d on X is **compatible** with the topology τ if τ is equal to the topology associated to the metric d.

If a topological space admits a compatible metric, we call it **metrizable**. Let us point out already that our main objects of study in this book, namely Polish spaces, are metrizable topological spaces.

Convention. We will often work with topological spaces whose topology will be implicit, i.e. we will only refer to elements of the topology as open sets. As we saw, the topology is also described by its associated neighorhood system, and we will also refer to it implicitly. For instance, the statement "Let X be a topological space, let $U \subseteq X$ be an open set, let $x \in U$ and let V be a neighborhood of x" means "Let (X, τ) be a topopological space and $(\mathcal{V}_x^\tau)_{x\in X}$ be the associated neighborhood system, let $U \in \tau$, let $x \in U$ and let $V \in \mathcal{V}_x^\tau$ ".

Exercise 1.4. Let X be a set.

1. Show that two metrics d_1 and d_2 on X induce the same topology if and only if for every $x \in X$ and every $\epsilon > 0$, there is $\delta > 0$ such that

$$B_{d_1}(x, \delta) \subseteq B_{d_2}(x, \epsilon) \text{ and } B_{d_2}(x, \delta) \subseteq B_{d_1}(x, \epsilon).$$

2. Show that if (X, d) is a metric space, then if we let $\tilde{d}(x, y) = \min(1, d(x, y))$, the map $\tilde{d}(x, y)$ is a metric compatible with the topology induced by d.

Remark 1.20. In Exercise **??** we will see a general recipe for building many compatible metrics from one.

1.2 Around convergence

1.2.1 Closed subsets and closures

Definition 1.21. Let (X, τ) be a topological space. A subset $F \subseteq X$ is **closed** if its complement is open, i.e. $X \setminus F \in \tau$.

We note the following reformulation of being closed in terms of neighborhoods systems, which we will very soon use without mention.

Proposition 1.22. *Let (X, τ) be a topological space. A subset $F \subseteq X$ is closed if and only if every $x \in X \setminus F$ has a neighborhood disjoint from F.*

Proof. By definition we have the following chain of equivalences: F is closed $\iff$ $X \setminus F$ is open $\iff$ $X \setminus F$ is a neighborhood of all its points $\iff$ every $x \in X \setminus F$ has a neighborhood contained in $X \setminus F$. But being contained in $X \setminus F$ is equivalent to being disjoint from F so F is closed if and only if every $x \in X \setminus F$ has a neighborhood disjoint from F. □

The collection of closed subsets of a topological space satisfies the following properties, dual to those of a topology[3]:

(a') $\emptyset$ and X are closed;

(b') any finite union of closed sets is closed;

[3] The proofs of these properties are straightforward consequences of the fact that taking the complement "exchanges unions and intersections".

(c') any intersection of closed sets is closed.

The notion of interior can be dualized by taking the complement: given a subset A of a topological space (X, τ), its **closure** $\overline{A}$ is the intersection of the closed sets containing A. Since any intersection of closed sets is closed, $\overline{A}$ is the smallest closed set containing A and A is closed if and only if $A = \overline{A}$.

Exercise 1.5. Let $\mathcal{V}$ be a neighborhood system on a set X. Show that given a subset $A \subseteq X$, the $\tau_{\mathcal{V}}$-closure of A is the set of all $x \in X$ such that every neighborhood of x intersects A[4].

Definition 1.23. Let X be a topological space. A subset $A \subseteq X$ is **dense** if $\overline{A} = X$.

Exercise 1.6. 1. Show that the set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$.

2. Show that $\mathbb{N}$ is dense in its one point compactification $\overline{\mathbb{N}}$ (see Def. 1.9), thus justifying the notation.

Let (X, d) be a metric space. The **closed ball** of radius $r \geqslant 0$ around $x \in X$ is defined by

$$B_d^{\leqslant}(x, r) := \{y \in X : d(x, y) \leqslant r\}.$$

Proposition 1.24. *Let (X, d) be a metric space, let $x \in X$ and $r > 0$. Then $B_d^{\leqslant}(x, r)$ is closed.*

Proof. We use the characterization of Prop. 1.22. Suppose $y \notin B^{\leqslant}(x, r)$, then $d(x, y) > r$. Then for every $z \in B_d(y, d(x, y) - r)$ we have

$$d(x, z) \geqslant |d(x, y) - d(y, z)| \geqslant d(x, y) - d(y, z) > d(x, y) - (d(x, y) - r)) > r,$$

so $B_d(y, d(x, y) - r)$ is a neighborhood of y which is still disjoint from $B_d^{\leqslant}(x, r)$, and we conclude that $B_d^{\leqslant}(x, r)$ is closed. □

When clear from context we simply write the closed ball of radius r around x as $B^{\leqslant}(x, r)$. Let us stress out that it is *not* equal to the closure of $B(x, r)$ in general, as opposed to what happens in normed vector spaces. A simple counterexample is given by open balls of radius 1 in the discrete metric defined in Example 1.18. We end this section by seeing how the distance to a set can yield both open and closed sets.

Exercise 1.7. Let (X, d) be a metric space, let $A \subseteq X$.

1. Show that $\overline{A} = \{x \in X : d(x, A) = 0\}$.

2. For $\epsilon > 0$, let the **ϵ-neighborhood** of A be the set

$$(A)_\epsilon := \{x \in A : d(x, A) < \epsilon\}.$$

Show that $(A)_\epsilon$ is open. (Hint: show that $(A)_\epsilon = \bigcup_{a \in A} B(a, \epsilon)$).

[4]In particular A is closed if and only if for every $x \in X$, one has $x \in A$ if and only if every neighborhood of x intersects A.

1.2.2 Convergence and Hausdorfness

Our main motivation for the introduction of neighborhood systems is that they provide a natural framework to recast the intuitive definition of convergence of a function in a more symmetric manner as explained right before Definition 1.2. The axiom of separation for metrics makes limits unique, and we need a replacement of it for neighborhood systems, or equivalently for topologies. Of the many possible choices, the following has proven to be the most robust.

Definition 1.25. A topological space (X, τ) is **Hausdorff** if for any $x, y \in X$ one has $x \neq y$ if and only if x and y have disjoint respective neighborhoods: there is a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.

Example 1.26. Metric spaces are Hausdorff, as witnessed by the open balls $B(x, \frac{d(x,y)}{2})$ and $B(y, \frac{d(x,y)}{2})$. Indeed, any element z of the intersection of these balls violates the triangle inequality. As a consequence, metrizable topological spaces are Hausdorff.

If (X, d) is a metric space, we saw that closed balls are closed. In particular, given $x \in X$, the closed ball of radius zero around x is closed, so $\{x\}$ is closed. The same is true in Hausdorff topological spaces.

Proposition 1.27. *Let X be a Hausdorff topological space. Then for every $x \in X$, the singleton $\{x\}$ is closed.*

Proof. Let $y \notin \{x\}$, then we find disjoint open sets U and V with $x \in U$ and $y \in V$. Then V is a neighborhood of y disjoint from $\{x\}$. This shows $\{x\}$ is closed by Prop. 1.22. □

We can now start studying convergence in general topological spaces. Let us first give the definition, which by now should not come as a surprise.

Definition 1.28. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. Given $x_0 \in X$, one says that $f(x)$ **tends to** y_0 as x tends to x_0 if the preimage under f of any neighborhood of y_0 is a neighborhood of x_0: for all $V \in \mathcal{V}_{y_0}^{\tau_Y}$, one has $f^{-1}(V) \in \mathcal{V}_{x_0}^{\tau_X}$.

Proposition 1.29. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$, $x_0 \in X$ and $y_0 \in Y$. Assume that Y is Hausdorff. If $f(x)$ tends to y_0 as x tends to x_0, then one must have $y_0 = f(x_0)$.*

Proof. Assume by contradiction that $f(x_0) \neq y_0$. By Hausdorfness, there are disjoint neighborhoods U of $f(x_0)$ and V of y_0. But then since both $f^{-1}(U)$ and $f^{-1}(V)$ are neighborhoods of x_0, their intersection $f^{-1}(U \cap V)$ is a neighborhood of x_0 hence non-empty, a contradiction. We conclude that $y_0 = f(x_0)$ as wanted. □

So when Y is Hausdorff we will write as in the metric case

$$\lim_{x \to x_0} f(x) = y_0 = f(x_0).$$

when $f(x)$ tends to y_0 as x tends to x_0.

1.2.3 Continuity

Definition 1.30. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. We say that f is continuous if for every $x_0 \in X$, one has that $f(x)$ tends to $f(x_0)$ as x tends to x_0.

Exercise 1.8. Show that the composition of any two continuous maps is continuous.

When we are dealing with metric spaces (X, d_X) and (Y, d_Y), observe that $f : X \to Y$ is continuous if and only if for every $x \in X$ and every $\epsilon > 0$, there is $\delta > 0$ such that for all x' satisfying $d_X(x, x') < \delta$ we have $d(f(x), f(x') < \epsilon$. We have the stronger notion of *uniform* continuity where given $\epsilon > 0$ we can now find a δ which works simultaneously for all $x \in X$.

Definition 1.31. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is **uniformly continuous** if for every $\epsilon > 0$ there is $\delta > 0$ such that for all $x, x' \in X$ satisfying $d_X(x, x') < \delta$, we have $d_Y(f(x), f(x')) < \epsilon$.

Finally, one has the even stronger condition of being Lipschitz.

Definition 1.32. Let (X, d_X) and (Y, d_Y) be metric spaces and $K > 0$. A map $f : X \to Y$ is K-**Lipschitz** if for all $x, x' \in X$, we have $d_Y(f(x), f(x')) \leqslant K d_X(x, x')$. When f is K-Lipschitz for some $K > 0$, we say that f is **Lipschitz**.

Example 1.33. Equation (1.2) from Exercise 1.1 can now be reformulated as: given any non-emtpy subset A of a metric space (X, d), the distance to A is a 1-Lipschitz function from (X, d) to $(\mathbb{R}, |\cdot|)$.

Observe that Lipschitz functions are indeed uniformly continuous: if $f : (X, d_X) \to (Y, d_Y)$ is K-Lipschitz, given $\epsilon > 0$, we have that $d_Y(f(x), f(x')) < \epsilon$ as soon as $d_X(x, x') < \epsilon/K$. Let us now come back to our topological framework and see how it provides an elegant equivalent definition for continuity.

Proposition 1.34. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. The following are equivalent.*

1. *The map f is continuous.*
2. *The preimage under f of any open set is open: for every $U \in \tau_Y$ we have $f^{-1}(U) \in \tau_X$.*
3. *The preimage under f of any closed set is closed: for every $F \subseteq Y$ closed in Y, we have $f^{-1}(F)$ closed in X.*

Proof. The equivalence between (2) and (3) is clear from the fact that f^{-1} preserves complements.

Let us prove that (2)⇒(1). Suppose that the preimage of any open set is open, and let $x_0 \in X$. If V is a neighborhood of $f(x_0)$, there is an open set U contained in V. The preimage of V is then open and contains x_0, so it is a neighborhood of x_0. We conclude that f is continuous.

Conversely, if f is continuous, let U be an open set and let $x_0 \in f^{-1}(U)$. Since U is open we find a neighborhood V of $f(x_0)$ contained in U. Since $f(x)$ tends to $f(x_0)$ as x tends to x_0, the preimage $f^{-1}(V)$ a neighborhood of x_0. Moreover $V \subseteq U$ implies $f^{-1}(V) \subseteq f^{-1}(U)$ so we conclude that $f^{-1}(U)$ is a neighborhood of x_0. So $f^{-1}(U)$ is a neighborhood of all its points hence open. □

We can now give a one line proof of the fact that given a set A, the set $\{x \in X : d(x, A) < \epsilon\}$ is open (question 2 from Exercise 1.7). Indeed $\{x \in X : d(x, A) < \epsilon\}$ is the preimage of the open set $]-\infty, \epsilon[$ via the distance to A map which is 1-Lipschitz hence continuous. Actually from now on we will follow the following principle:

In order to show that a set is closed (resp. open), show that it is the preimage via a continuous map of a set which is already known to be closed (resp. open) !

Of course for this to work one needs a bunch of open and closed sets to start with. Open and closed balls are a start, but this method will really prove efficient once we have the definition of the product topology and obtain some natural continuous functions and closed sets from this construction (namely projections and diagonals, see Sec. 1.3.3).

1.2.4 Convergence of partially defined functions

We will often need to deal with functions which are not defined everywhere, i.e. of the form $f : A \to Y$ where $A \subseteq X$ and the limit point x_0 may not belong to A. Convergence of $f(x)$ when x tends to x_0 is then defined in a similar way as before by considering the intersections of the neighborhoods of x_0 with A. For this to make sense, we need these neighborhoods to actually intersect A, i.e. we need that $x_0 \in \overline{A}$. Note that however we don't necessarily have $x_0 \in A$.

Definition 1.35. Let X be a topological space, let $A \subseteq X$ and let $x_0 \in \overline{A}$. Let Y be a topological space, let $y_0 \in Y$ and $f : A \to Y$.

We say that $f(x)$ tends to y_0 as x tends to x_0 if whenever V is a neighborhood of y_0, there exists a neighborhood W of x_0 such that $f^{-1}(V) = W \cap A$.

Let us remark that the above definition is compatible with the one where $A = X$ (Def. 1.28). The following proposition, although simple, is important.

Proposition 1.36. *Let X be a topological space, let $A \subseteq X$ and let $x_0 \in \overline{A}$. Let Y be a topological space, let $y_0 \in Y$ and $f : A \to Y$. If $f(x)$ tends to y_0 as x tends to x_0 then $y_0 \in \overline{f(A)}$.*

Proof. Let V be a neighborhood of y_0, then by definition $f^{-1}(V)$ is of the form $A \cap W$ for some neighborhood W of x_0. Since $x_0 \in \overline{A}$ we have $A \cap W \neq \emptyset$ so $f^{-1}(V)$ is nonempty. This means that V intersects $f(A)$ and thus every neighborhood of y_0 intersects $f(A)$. We conclude that $y_0 \in \overline{f(A)}$. □

Exercise 1.9. Let X, Y, Z be topological spaces, $A \subseteq X$ and $B \subseteq Y$. Let $f : A \to B$ and $g : B \to Z$. Show that if $f(x)$ tends to y_0 as x tends to x_0 and if $g(y)$ tends to $g(y_0)$ as y tends to y_0 thn $f \circ g(x)$ tends to z_0 as x tends to x_0

The following proposition establishes the uniqueness of the limit when it exists.

Proposition 1.37. *If Y is Hausdorff, then there is at most one $y_0 \in Y$ such that $f(x)$ tends to y_0 as x tends to x_0.*

Proof. Suppose $f(x)$ tends to both y_0 and y_1 as x tends to x_0. By Hausdorffness, there are disjoint neighborhoods U of x_0 and V of y_1. But then since both $f^{-1}(U)$ and $f^{-1}(V)$ are neighborhoods of x_0, their intersection $f^{-1}(U \cap V)$ must be nonempty, a contradiction. We conclude that $y_0 = y_1$ hence limit points are unique. □

When Y is Hausdorff, we will thus write

$$\lim_{x \to x_0} f(x) = y_0.$$

Since we asked in the above definition that $x_0 \in \overline{A}$, let us remark that $y_0 \in \overline{f(A)}$. Indeed for every neighborhood V of y_0 we have $f^{-1}(V) = W \cap A$ for some neighborhood W of x_0 and since $x_0 \in \overline{A}$ we have $W \cap A \neq \emptyset$ so $V \neq \emptyset$.

We continue with an important property of continuous functions: they are uniquely determined by their restriction to a dense subset.

Proposition 1.38. *Let X be a topological space and let Y be a Hausdorff topological space. Let A be a dense subset of X. Then every continuous function $f : X \to Y$ is completely determined by its restriction to A: if $g : X \to Y$ is another continuous function such that $f_{\restriction A} = g_{\restriction A}$ then $f = g$.*

Proof. Let $x_0 \in X$, then $f(x_0)$ is the limit as x tends to x_0 of $f(x)$. Since $x \in \overline{A}$ we can also consider the limit of the restriction of f to A as x tends to x_0. A direct application of the definition then shows that $f_{\restriction A}(x)$ tends to $f(x_0)$ as x tends to x_0. By the same argument $g_{\restriction A}(x)$ tends to $g(x_0)$ as x tends to x_0.

Since $g_{\restriction A} = f_{\restriction A}$ the uniqueness of the limit that we just proved implies $f(x_0) = g(x_0)$. □

Let us now apply the notion of convergence to the point $+\infty$ in the topological space $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Proposition 1.39. *Let (X, τ) be a topological space and let $(u_n)_{n \in \mathbb{N}}$ be a sequence of elements of x and let $l \in X$. Consider the map $\tilde{u} : \mathbb{N} \cup \{+\infty\} \to X$ defined by $\tilde{u}(n) = u_n$ and $\tilde{u}(+\infty) = l$. The following are equivalent:*

(i) $\lim_{n \to +\infty} u_n = l$

(ii) for all neighborhood V of l there is $N \in \mathbb{N}$ such that for every $n \geqslant N$ we have $u_n \in V$.

(iii) the map $\tilde{u}$ is continuous.

Moreover, if one of the above conditions is satisfied, then $l \in \overline{u_n : n \in \mathbb{N}}$.

Proof. The equivalence between (i) and (ii) is a direct application of the definitions convergence (Def. 1.35) and of the neighborhoods of $+\infty$ (Def. 1.9).

So we need to show that (i) $\iff$ (iii). Note that for every $n \in \mathbb{N}$, the set $\{n\}$ is a neighborhood of n so we always have $\lim_{x \to n} \tilde{u}(x) = \tilde{u}(n) = u_n$. So saying that $\tilde{u}$ is continuous means that $\lim_{x \to +\infty} \tilde{u}(x) = \tilde{u}(+\infty)$, i.e. $\lim_{n \to +\infty} u_n = l$. □

Exercise 1.10. Let $f : X \to Y$ be a continuous map where Y is Hausdorff. Show that for every sequence (x_n), if $\lim\limits_{n \to +\infty} x_n = x$ then $\lim\limits_{n \to +\infty} f(x_n) = f(x)$. We will now see that one can characterize continuity this way when Y is moreover *first-countable*.

1.2.5 First-countability and sequences

Definition 1.40. Let (X, τ) be a topological space and let $x \in X$. A family $\mathcal{V}$ of neighborhoods of x is a **neighborhood basis** for x if for every neighborhood W of x, there is $V \in \mathcal{V}$ such that $V \subseteq W$.

Example 1.41. In a metric space (X, d), the sequence $(V_n)_{n \in \mathbb{N}^*}$ defined by $V_n := B(x, \frac{1}{n})$ is a countable neighborhood basis of x such that $\overline{V}_{n+1} \subseteq V_n$.

Definition 1.42. A topological space (X, τ) is **first-countable** if all its elements have a countable neighborhood basis.

Example 1.43. As an immediate consequence of Example 1.41, metrizable topological spaces are first-countable.

Here is a useful observation about first-countability which provides something similar to balls of radius $1/n$ in the metrizable case.

Lemma 1.44. *Let X be a topological space, suppose that $x \in X$ has a countable neighborhood basis. Then x has a neighborhood basis $(V_n)_{n \in \mathbb{N}}$ which is decreasing, i.e. such that for all $n \in \mathbb{N}$ we have $V_{n+1} \subseteq V_n$.*

Proof. Let $(W_n)_{n \in \mathbb{N}}$ be a neighborhood basis of x and put $V_n = \bigcap_{i=0}^{n} W_i$. By construction $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$ and each V_n is a neighborhood of x as desired. □

The main good thing about first-countability is that it allows one to think about many topological concept in terms of sequences.

Proposition 1.45. *Let X be a metrizable or more generally first-countable topological space and let $A \subseteq X$. Then the closure $\overline{A}$ of A is the set of limits of sequences of elements of A:*

$$\overline{A} = \{x \in X : \exists (x_n) \in A^{\mathbb{N}}, x_n \to x\}.$$

In particular, A is closed if and only if every convergent sequence of elements of A converges to an element of A.

Proof. Let us first show that every limit of sequence of elements of A belongs to $\overline{A}$ (this does not use the fact that X is first-countable and also follows from). Suppose $x = \lim x_n$ with $x_n \in A$ and V is a neighborhood of x. Since $x = \lim x_n$ there is $n \in \mathbb{N}$ such that $x_n \in V$ and hence $V \cap A \neq \emptyset$. So $x \in \overline{A}$.

Conversely let $x \in \overline{A}$, then for every neighborhood V of x, one has $V \cap A \neq \emptyset$. Let (V_n) be a decreasing neighborhood basis of x as per Lem. 1.44. For every $n \in \mathbb{N}$ we find $x_n \in V_n \cap A$. We then have $x_n \to x$ as wanted. Indeed if V is a neighborhood of x there is $N \in \mathbb{N}$ such that $V_N \subseteq V$ since (V_n) is a basis. Since (V_n) is decreasing for all $n \geqslant N$ we also have $V_n \subseteq V$ and hence $x_n \in V$.

For the "in particular" part of the proposition, recall that A is closed if and only if $A = \overline{A}$. □

Proposition 1.46. *Let X and Y be topological spaces, let $f : X \to Y$ and suppose X is metrisable or more generally first-countable. Then the map f is continuous if and only if for every $x \in X$ and every sequence $x_n \to x$, one has $f(x_n) \to f(x)$.*

Proof. The direct implication was done in Exercise 1.10: if f is continuous then $x_n \to x$ implies $f(x_n) \to f(x)$.

The converse will be proved via the characterization of continuity in terms of preimages of closed sets being closed (item (3) in Prop. 1.34) and the previous proposition. Let F be a closed subset of Y, let us show that $f^{-1}(F)$ is closed. By the previous proposition it suffices to show that every convergent sequence of elements of F converges to an element of F. So let $x_n \to x$ for some $x \in X$ where $x_n \in f^{-1}(F)$ for all $n \in \mathbb{N}$. By definition for all $n \in \mathbb{N}$ we have $f(x_n) \in F$, and by assumption $f(x_n) \to f(x)$. We conclude that $f(x) \in F$, i.e. $x \in f^{-1}(F)$ which is thus a closed set as wanted. □

Remark 1.47. Using sequences to check continuity is often quite convenient since we are accustomed to dealing with convergence. The notion of *net* provides a useful replacement in the general case, see exercise **??**.

1.2.6 Homeomorphisms

Homeomorphisms are fundamental because many of the properties that we deal with are not changed when passing to a homeomorphic space. So rather than working with our original topological space X, it will often be convenient to work with another space Y homeomorphic to it with some nice additional features.

Definition 1.48. Let X and Y be two topological spaces. A map $f : X \to Y$ is called a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous. When there is a homeomorphism $X \to Y$, we say that X and Y are **homeomorphic**.

A map $f : X \to Y$ is called a **homeomorphism onto its image** or an **embedding** if it is injective and if its corestriction $X \to f(X)$ is a homeomorphism (where $f(X)$ is equipped with the induced topology).

Observe that X is homeomorphic to Y if and only Y is homeomorphic to X. Let us make more precise the assertion that homeomorphisms preserve the topology.

Exercise 1.11. Let (X, τ_X) and (Y, τ_Y) be topological spaces. Show that $f : X \to Y$ is a homeomorphism if and only if f is continuous bijective and $\tau_Y = \{f(U) : U \in \tau_X\}$.

All the topological properties that we will see are invariant under homeomorphism (which means that if X satisfies a property and Y is homeomorphic to X then Y also does). As an example, let us spell out why Hausdorfness is invariant under homeomorphism. The idea is simply to use the homeomorphism to transport relevant objects between X and Y.

Proposition 1.49. *Let X and Y be two homeomorphic topological spaces. If X is metrizable then so is Y. If X is Hausdorff then so is Y.*

Proof. Let $f : X \to Y$ be a homeomorphism. Assume X is Hausdorff and let $f : X \to Y$ be a homeomorphism. Let $y_1 \neq y_2 \in Y$. Then $f^{-1}(y_1) \neq f^{-1}(y_2)$ so there are disjoint open sets U_1 and U_2 such that $f^{-1}(y_1) \in U_1$ and $f^{-1}(y_2) \in U_2$. Since f is a homeomorphism $f(U_1)$ and $f(U_2)$ are disjoint open sets, and $y_1 \in f(U_1)$ while $y_2 \in f(U_2)$: Y is Hausdorff as promised. □

Remark 1.50. See Exercise 1.36 for other properties which are invariant under homeomorphism.

Let us end this section by giving a sequential characterization of homeomorphisms similar to the one of continuity from the previous section.

Proposition 1.51. *Let X and Y be metrisable or more generally first-countable Hausdorff topological spaces. Then a map $f : X \to Y$ is a homeomorphism onto its image if and only if the following condition is satisfied: for every sequence $(x_n) \in X^{\mathbb{N}}$ and every $x \in X$ we have $x_n \to x \iff f(x_n) \to f(x)$.*

Proof. The direct implication is clear from Prop. 1.46.

Now assume conversely that for every sequence $(x_n) \in X^{\mathbb{N}}$ and every $x \in X$ we have $x_n \to x \iff f(x_n) \to f(x)$.

Let us first check that f is injective: if $f(x) = f(y)$ we consider the constant sequence $x_n = x$ and we thus have $f(x) \to f(y)$ as n tends to $+\infty$. So by assumption we have $x \to y$ as n tends to $+\infty$; since X is Hausdorff this implies $x = y$.

The continuity of f and $f^{-1} : f(X) \to X$ now follows from the previous proposition, and we conclude f is a homeomorphism onto its image. □

1.3 Operations on topological spaces

The purpose of this section is to see how to build new topological spaces out of old ones. These constructions will in turn provides ways to build new Polish spaces in the next chapter.

1.3.1 Induced topology

Let us start by the most basic such construction: given a topological space X and a subset $A \subseteq X$, there is a natural way to equip A with a topology.

Definition 1.52. The **induced topology** τ_A on a subset A of a topological space (X, τ_A) is defined by

$$\tau_A = \{U \cap A : U \in \tau\}.$$

In other words, a set $U \subseteq A$ is τ_A-open if and only if it is the intersection with A of a τ-open subset of X.

Exercise 1.12. Let (X, τ) be a topological space, and let A be a subset of X.

1. Check that the induced topology τ_A is a topology, and show that it is given by the following neighborhood system: given $x \in A$, a subset V of A is a neighborhood of x if and only if it is the intersection with A of a τ-neighborhood of x.

2. Let $f : A \to Y$ be a map where Y is a topological space, and let $x_0 \in A$. Prove that the definitions of the convergence of $f(x)$ as x tends to x_0 are the same wether we chose to view f as a partially defined map (as in Def. 1.35) or as a map defined on the topological space (A, τ_A) (as in Def. 1.28).

3. Let $f : X \to Y$ be a map, where Y is a topological space. Prove that f is continuous if and only if its corestriction $\overline{f} : X \to f(X)$ is continuous, where $f(X)$ is equipped with the induced topology.

1.3.2 Disjoint union topology

We give the second construction in terms of neighborhood systems.

Definition 1.53. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The **disjoint union topology** is the topology on $X \sqcup Y$ associated to the **disjoint union neighborhood system** defined by: given $x \in X \sqcup Y$, a subset $V \subseteq X \sqcup Y$ containing is a x neighborhood of x when either

- $x \in X$ and $V \cap X$ is a τ_X-neighborhood of x or
- $x \in Y$ and $V \cap Y$ is a τ_Y-neighborhood of x.

Remark 1.54. The disjoint union topology is sometimes called the *direct sum* topology for category theoretic reasons (cf. question 2 of the following exercise).

Exercise 1.13. Let (X, τ_X) and (Y, τ_Y) be two topological spaces.

1. Check that the disjoint union neighborhood system is indeed a neighborhood system.
2. Show that if Z is a topological space and $f_X : X \to Z$, $f_Y : Y \to Z$ are continuous maps, then the map $f : X \sqcup Y \to Z$ defined by

$$f(x) = \begin{cases} f_X(x) & \text{if } x \in X \\ f_Y(x) & \text{if } x \in Y \end{cases}$$

is continuous for the disjoint union topology, and that conversely every continuous map $f : X \sqcup Y \to Z$ can be decomposed likewise.

3. Show that a subset $U \subseteq X \sqcup Y$ is open if and only if its intersection with X is τ_X-open while its intersection with Y is τ_Y-open.
4. Generalize this construction to arbitrary disjoint unions.

1.3.3 Product topology

We now finally move to the fundamental concept of the product topology which can be summed up by the following slogan: "convergence in the product topology is equivalent to convergence in each coordinate" (see Prop. 1.58).

We will present the construction in terms of neighborhood systems and leave the topological formulation as an exercise.

If $(X_i)_{i \in I}$ is family of sets then for each $j \in I$ we denote by π_j the projection $\prod_{i \in I} X_i \to X_j$ defined by $\pi_j((x_i)_{i \in I}) = x_j$.

Observe that if we want convergence in the product topology on $\prod_{i \in I} X_i$ to *imply* convergence on some coordinate $i_0 \in I$, we need that for every $(x_i)_{i \in I} \in \prod_{i \in I} X_i$ and every neighborhood V_{i_0} of x_{i_0}, the set $\pi_{i_0}^{-1}(V_{i_0})$ is a neighborhood of x. The fact that finite intersections of neighborhoods are neighborhoods then forces us to make the following definition where neighborhoods in the product put constraints on finitely many coordinates.

Definition 1.55. Let $(X_i)_{i\in I}$ be a family of topological spaces. Then the **product neighborhood system** on $\prod_{i\in I} X_i$ is defined as follows: a set $V \subseteq \prod_{i\in I} X_i$ is a neighborhood of $(x_i)_{i\in I}$ if and only if it contains a set of the form

$$\bigcap_{i\in K} \pi_i^{-1}(V_i)$$

for some finite subset K of I, where for each $i \in K$ the set V_i is a neighborhood of x_i. The associated topology is called the **product topology**.

Let us note the following reformulation: in the product neighborhood system, a set $V \subseteq \prod_{i\in I} X_i$ is a neighborhood of $(x_i)_{i\in I}$ if and only if it contains a set of the form

$$\{(y_i)_{i\in I} : y_{i_1} \in V_{i_1} \text{ and ... and } y_{i_n} \in V_{i_n}\}$$

for some $n \in \mathbb{N}$ and some distinct $i_1, ..., i_n \in I$, where for each $k = 1, ..., n$ the set V_{i_k} is a neighborhood of x_{i_k}. In more informal terms, an element y of the product is close to x when it has finitely many coordinates close to those of x. Let us now check that the product neighborhood system is indeed a neighborhood system.

Proof that the product neighborhood system is a neighborhood system. Conditions (i), (iii) and (iv) are clear from the definition. The stability under finite intersection is also not hard to see: if V and V' are neighborhoods of x, then V contains $\bigcap_{i\in K}^n \pi_i^{-1}(V_i)$ and W contains $\bigcap_{i\in L} \pi_i^{-1}(V_i')$ where $(V_i)_{i\in K}$ and $(V_i')_{i\in L}$ are two finite families of neighborhoods. For $i \in K \cup L$, we let

$$W_i = \begin{cases} V_i \cap V_i' & \text{if } i \in K \cap L \\ V_i & \text{if } x \in K \setminus L \\ V_i' & \text{if } x \in L \setminus K \end{cases}$$

Note that each W_i is still a neighborhood of x_i. Then $V \cap W$ contains

$$\bigcap_{i\in K\cup L} \pi_i^{-1}(W_i),$$

witnessing that $V \cap W$ is a neighborhood of $(x_i)_{i\in I}$.

Finally let us check that property (v) is satisfied: let V be a neighborhood of $(x_i)_{i\in I}$, and let $(V_i)_{i\in K}$ neighborhoods such that $\bigcap_{i\in K} \pi_i^{-1}(V_i) \subseteq V$. Then for each $i \in K$ we have $W_i \subseteq V_i$ such that V_i is a neighborhood of every element of W_i and W_i is a neighborhood of x_i. Then $\bigcap_{i\in K} \pi_i^{-1}(W_i)$ is a neighborhood of (x_i) and $\bigcap_{i\in K} \pi_i^{-1}(V_i)$ (hence V) is a neighborhood every $(y_i) \in \bigcap_{i\in K} \pi_i^{-1}(W_i)$. □

Exercise 1.14. Show that if $(X_n)_{n\in\mathbb{N}}$ is a sequence of first-countable topological spaces, then the product $\prod_{n\in\mathbb{N}} X_n$ is first-countable.

Remark 1.56. When we are dealing with a finite product of topological spaces $\prod_{i=1}^n X_i$, a basis of neighborhoods of $(x_i)_{i=1}^n$ is simply given by the set of $V_1 \times \cdots \times V_n$ where each V_i is a neighborhood of x_i.

An important fact about Hausdorff topological spaces is that equality defines a closed relation for the product topology.

Proposition 1.57. *Let X be a Hausdorff topological space. Then the diagonal subspace $\Delta_X = \{(x, y) \in X^2 : x = y\}$ is closed in X^2.*

Proof. We prove that the complement of Δ_X is open: if $x \neq y$, by Hausdorfness x and y have disjoint neighborhoods U and V, so $U \times V$ is a neighborhood of (x, y) disjoint from Δ_X. □

Exercise 1.15. Prove more generally that if X is Hausdorff and I is a set, then the space of constant maps $I \to X$ is closed in X^I. (Hint: write it as an intersection of closed sets).

The following characterization of convergence in product spaces is fundamental.

Proposition 1.58. *Let X be a topological space, let $A \subseteq X$ and let $(Y_i)_{i\in I}$ be a family of topological spaces. Consider a map $f : A \to \prod_{i\in I} Y_i$. Given $x_0 \in \overline{A}$ and $(y_i)_{i\in I} \in \prod_{i\in I} Y_i$ the following are equivalent:*

(i) $f(x)$ tends to $(y_i)_{i\in I}$ as x tends to x_0;

(ii) for all $i \in I$, $\pi_i(f(x))$ tends to y_i as x tends to x_0.

Proof. Assuming (i), we get (ii) because π_i is continuous and convergence behaves well with respect to composition (see Exercise 1.9).

Conversely assume (ii): for all $i \in I$, $\pi_i(f(x))$ tends to y_i as x tends to x_0. Let $V \supseteq \bigcap_{i\in K} \pi_i^{-1}(V_i)$ be a neighborhood of $(y_i)_{i\in I}$, where K is finite and for $i \in K$ the set V_i is a neighborhood of y_i. Observe that

$$f^{-1}\left(\bigcap_{i\in K} \pi_i^{-1}(V_i)\right) = \bigcap_{i\in K} f^{-1}\pi_i^{-1}(V_i) = \bigcap_{i\in K} (\pi_i \circ f)^{-1}(V_i).$$

Since for each $i \in K$ we have $\pi_i(f(x)) \to y_i$ as x tends to x_0, each $(\pi_i \circ f)^{-1}(V_i)$ is a neighborhood of x_0, and we conclude that $f^{-1}(V)$ is a neighborhood of x_0. So (i) holds as wanted. □

If $(X_n, d_n)_{n\in\mathbb{N}}$ are metric spaces, there is no canonical notion of a "product metric" on the product space $\prod_{n\in\mathbb{N}} X_n$ (even for finite products). However, we can always find a metric compatible with the product topology. Let us start with the easier finite case.

Exercise 1.16. Let (X_1, d_1),..., (X_n, d_n) be metric spaces. Show that the metric d on $\prod_{i=1}^n X_i$ defined by $d((x_1, ..., x_n), (y_1, ..., y_n)) = \sum_{i=1}^n d_i(x_i, y_i)$ is a metric compatible with the product topology.

In the infinite case, we will use the same idea of taking a sum, but the sum is indexed over an infinite set so we need to make sure it converges. The trick is to replace each metric d_n by $\frac{1}{2^n}\min(1, d_n)$.

Proposition 1.59. *Suppose $(X_n)_{n\in\mathbb{N}}$ is a countable family of metrizable topological spaces. Then the product topology on $\prod_{n\in\mathbb{N}} X_n$ is metrizable.*

Moreover, if for each n we have a metric d_n compatible with the topology of X_n, then the metric

$$d((x_n), (x'_n)) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \min(1, d_n(x_n, x'_n)).$$

is compatible with the product topology on $\prod_n X_n$.

Proof. The first part of the proposition clearly follows from the second, so we only need to show that the function d defined in the proposition is a metric compatible with the product topology.

Noting that for each $n \in \mathbb{N}$, the map $(x, x') \mapsto \min(1, d_n(x, x'))$ is a metric on X_n compatible with the topology induced by d_n (see Exercise 1.4), it is straightforward to check that d is a metric.

To see that d is compatible with the product topology, we need to show that the identity map on $\prod_{n\in\mathbb{N}} X_n$ is a homeomorphism when on one side we put the product topology and on the other side the topology induced by d. Since $\prod_{n\in\mathbb{N}} X_n$ is first-countable for the product topology (Exercise 1.14), by Prop. 1.51 we only need to check that for a sequence $((x^n_m)_{m\in\mathbb{N}})_{n\in\mathbb{N}}$ of elements of $\prod_n X_n$ and $(x'_m)_{m\in\mathbb{N}} \in \prod_n X_n$, we have $((x^n_m)_{m\in\mathbb{N}})_{n\in\mathbb{N}} \to (x'_m)_{m\in\mathbb{N}}$ in the product topology as n tends to $+\infty$ iff $d(((x^n_m), (x'^n)) \to 0$ as n tends to $+\infty$.

But by the dominated convergence theorem for series, $d(((x^n_m), (x'_m)) \to 0$ is equivalent to $d_m(x^n_m, x'_m) \to 0$ for all $m \in \mathbb{N}$, which by the previous proposition is in turn equivalent to $((x^n_m)_{m\in\mathbb{N}})_{n\in\mathbb{N}} \to (x'_m)_{m\in\mathbb{N}}$ in the product topology as wanted. □

1.3.4 Projective limits

We won't need inverse limits very often, so the reader may skip this part and get back to it when needed.

Definition 1.60. Let $(I, <)$ be an ordered set, meaning that for all $i, j, k \in I$ we have $i \not< i$ and the implication

$$i < j \text{ and } j < k \Rightarrow i < k.$$

Suppose moreover that I is **directed**, which means that for every $i, j \in I$ there is $k \in I$ such that $k \geqslant i$ and $k \geqslant j$. A couple $((X_i)_{i\in I}, (f_{i,j})_{i<j})$ is a **projective diagram** of topological spaces when the following conditions are satisfied:

- $(X_i)_{i\in I}$ is a family of Hausdorff topological spaces;
- for each $i < j$, we have that $f_{i,j} : X_j \to X_i$ is continuous;
- for each $i < j < k$ we have $f_{i,k} = f_{i,j} \circ f_{j,k}$.

Definition 1.61. The **projective limit** (or limit or inverse limit) of a projective diagram of topological spaces $((X_i)_{i\in I}, (f_{i,j})_{i<j})$ is denoted by $\varprojlim X_i$ and defined by

$$\varprojlim X_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i : \forall i < j, f_{i,j}(x_j) = x_i \right\}.$$

It is equipped with the topology induced by the product topology on $\prod_{i\in I} X_i$.

Example 1.62. Let $p \in \mathbb{N}$ be a prime number. For each $n < m \in \mathbb{N}$, we have $p^m\mathbb{Z} \leqslant p^n\mathbb{Z}$ so we have a surjection $f_{n,m} : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. By putting the discrete topology on each $\mathbb{Z}/p^n\mathbb{Z}$, we get a projective diagram. The associated projective limit is the space $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ which has a natural ring structure and is called the **ring of p-adic integers**. See Exercise **??** for more on this.

Projective limits satisfy the following universal property.

Proposition 1.63. *Let $((X_i)_{i\in I}, (f_{i,j})_{i<j})$ be a projective diagram of topological spaces. Whenever Y is a topological space endowed with continuous maps $g_i : Y \to X_i$ such that for each $i < j$ we have $f_{i,j}g_j = g_i$, then there is a* unique *continuous map $g : Y \to \varprojlim X_i$ satisfying $\pi_i g = g_i$.*

Proof. Observe that the condition $\pi_i g = g_i$ means that all the coordinates of the map $g : Y \to \prod X_i$, are prescribed, so uniqueness is clear and we must define g by $g(y) = (g_i(y))_{i\in I}$. The continuity of g is then a consequence of the continuity of each g_i and of the characterization of convergence in product spaces (Prop. 1.58). □

Proposition 1.64. *The projective limit of any projective diagram of topological spaces $((X_i)_{i\in I}, (f_{i,j})_{i<j})$ is a closed subspace of the product space $\prod_{i\in I} X_i$.*

Proof. This is left to the reader, who can either do it by hand (i.e. show directly that its complement is open) or use the more “descriptive” way of looking at it suggested by the next exercise. □

Exercise 1.17. Show that for each $i < j$, the map $(x_i)_{i\in I} \mapsto (f_{i,j}(x_j), x_i)$ is continuous. Deduce that the set $F_{i,j} = \left\{(x_i)_{i\in I} \in \prod_{i\in I} X_i : f_{i,j}(x_j) = x_i\right\}$ is closed (hint: use Prop. 1.57). Use this to prove Prop. 1.64.

An easy way for an ordered set to be directed is to have a maximum. However in this case the resulting topological space is not interesting as the following exercise shows.

Exercise 1.18. Let $((X_i)_{i\in I}, (f_{i,j})_{i<j})$ be a projective diagram of topological spaces, suppose that we have $k \in I$ such that for all $i \in I$, $i \leqslant k$. Show that the projective limit $\varprojlim X_i$ is homeomorphic to X_k. Why is this a direct consequence of the next proposition ?

Let $(I, <)$ be a directed set, a subset $K \subseteq I$ is **cofinal** if for every $i \in I$ there is $k \in K$ such that $i < k$. Observe that if K is cofinal, then it is directed as well. In particular when $((X_i)_{i\in I}, (f_{i,j})_{i<j})$ is a projective diagram of topological spaces and K is cofinal in I then $((X_k)_{k\in K}, (f_{k,l})_{k<l})$ also is. The next proposition shows that the projective limit is the same.

Proposition 1.65. *Let $(I, <)$ be a directed set, let $K \subseteq I$ be cofinal. Let $((X_i)_{i\in I}, (f_{i,j})_{i<j})$ be a projective diagram of topological spaces, then the projection map $\prod_{i\in I} X_i \to \prod_{k\in K} X_k$ induces a homeomorphism between the inverse limits $\varprojlim_{i\in I} X_i$ and $\varprojlim_{k\in K} X_k$.*

1.4 Generating a topology

We have so far presented topologies in terms of neighborhoods systems because we believe this notion is more natural to grasp when one is familiar with convergence. However, the elegance of the three axioms of topologies (namely (a) containing the empty set and the whole space, (b) being stable under finite intersections and (c) being stable under arbitrary reunions) makes them more flexible. In particular, it allows one to give very concise definitions of the above constructions.

Let us start with an easy operation on topologies on a common set X. Recall that $\mathcal{P}(\mathcal{P}(X))$ is an ordered set for inclusion, and since topologies are elements of the latter, we can compare them and say a topology τ is greater than another topology τ' when $\tau' \subseteq \tau$[5].

Lemma 1.66. *Let $(\tau_i)_{i\in I}$ be a non-empty family of topologies on a set X. Then $\bigcap_{i\in I}\tau_i$ is a topology. It is the greatest topology contained in each τ_i.*

Proof. That $\bigcap_{i\in I}\tau_i$ is a topology is a direct consequence of the fact that each τ_i is:

(a) for each $i \in I$ we have $\emptyset, X \in \tau_i$ so $\emptyset, X \in \bigcap_{i\in I}\tau_i$;

(b) if $U, V \in \bigcap_{i\in I}\tau_i$ then for each $i \in I$ we have $U, V \in \tau_i$ so $U \cap V \in \tau_i$ because τ_i is a topology. We conclude that $U \cap V \in \bigcap_{i\in I}\tau_i$;

(c) similarly if $(U_j)_{j\in J}$ is an arbitrary family of elements of $\bigcap_{i\in I}\tau_i$, then for each $i \in I$ we have $\bigcup_{j\in J} U_j \in \tau_i$ because the latter is a topology and hence $\bigcup_{j\in J} U_j \in \bigcap_{i\in I}\tau_i$.

If $(U_j)_{j\in J}$ is an arbitrary family of elements of $\bigcap_{i\in I}\tau_i$, then for each $i \in I$ we have $\bigcup_{j\in J} U_j \in \tau_i$ because the latter is a topology and hence $\bigcup_{j\in J} U_j \in \bigcap_{i\in I}\tau_i$. The set $\bigcap_{i\in I}\tau_i$ is the greatest set contained in each τ_i, and since it also happens to be a topology it must be the greatest topology contained in each τ_i. □

Definition 1.67. Let X be a set, let $\mathcal{A} \subseteq \mathcal{P}(X)$. The **topology generated** by $\mathcal{A}$ is the smallest topology containing $\mathcal{A}$, denoted by $\tau(\mathcal{A})$.

To see why this definition makes sense, consider the set $\mathcal{T}_\mathcal{A}$ of all the topologies containing $\mathcal{A}$. Such a set is non-empty since it contains $\mathcal{P}(X)$. Then $\tau(\mathcal{A}) := \bigcap_{\tau\in\mathcal{T}_\mathcal{A}}\tau$ satisfies the requirements of the definition: it is by construction smaller than any topology containing $\mathcal{A}$, and it is a topology by the above lemma.

When $\tau = \tau(\mathcal{A})$ we say that $\mathcal{A}$ is a **subbasis** for the topology τ. Here is an important application.

Proposition 1.68. *Let X and Y be topological spaces. Suppose $\mathcal{A}$ is a subbasis for the topology of Y. Then $f : X \to Y$ is continuous if and only if for every $U \in \mathcal{A}$, the set $f^{-1}(U)$ is open in X.*

Proof. Denote by τ_X the topology of X and by τ_Y the topology of Y. Let $\tau' := \{U \in \tau_Y : f^{-1}(U) \in \tau_X\}$. By our assumptions τ' contains $\mathcal{A}$, and since τ_X contains X and $\emptyset$ we have that τ' contains Y and $\emptyset$. Moreover τ' is stable under arbitrary reunions: if $(U_i)_{i\in I}$ is a family of open subsets of Y such that for every $i \in I$ we have $f^{-1}(U_i) \in \tau_X$, then $f^{-1}(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f^{-1}U_i \in \tau_X$. So τ' is a topology which contains $\mathcal{A}$, and since the latter is a subbasis for τ_Y, we conclude $\tau_Y \subseteq \tau'$. By definition this means that f is continuous. □

Let us now see a more concrete way of building $\tau(\mathcal{A})$. We make the convention that the intersection of an empty family of subsets of X is equal to X, and that the union of an empty family of sets is empty.

Proposition 1.69. *Let X be a set, let $\mathcal{A} \subseteq \mathcal{P}(X)$. Then the topology $\tau(\mathcal{A})$ generated by $\mathcal{A}$ is the set of unions of finite intersections of elements of $\mathcal{A}$.*

[5] When a topology τ contains a topology τ', one sometimes says that τ refines τ' or that τ' is coarser than τ. We believe however that these terms can add confusion and thus will not use them in this book.

Proof. Let us check that if we define $\tau'(\mathcal{A})$ as the set of unions of finite intersections of elements of $\mathcal{A}$, then $\tau'(\mathcal{A})$ is the smallest topology containing $\mathcal{A}$. Let us first see why $\tau'(\mathcal{A})$ is a topology. The verification is tedious but straightforward.

Clearly $\tau'(\mathcal{A})$ contains $\mathcal{A}$. Let $\mathcal{B}$ be the set of finite intersections of elements of $\mathcal{A}$, then $\tau'(\mathcal{A})$ is the set of reunions of elements of $\mathcal{B}$ and hence contains $\mathcal{B}$. By definition $B \in \mathcal{B}$ if and only if there is $F \subseteq \mathcal{A}$ finite such that

$$B = \bigcap_{A \in F} A.$$

If we take $F = \emptyset$ we get that $X \in \mathcal{B} \subseteq \tau(A)$, and since a reunion over the empty set is empty we see that $\emptyset \in \tau'(A)$. This takes care of axiom (a).

Let us now move to axiom (b), which is stability under finite intersections. Observe that $\mathcal{B}$ is stable under finite intersections because if $F, F' \subseteq \mathcal{A}$ are finite then

$$\bigcap_{A \in F} A \cap \bigcap_{A \in F'} A = \bigcap_{A \in F \cup F'} A$$

and $F \cup F'$ is finite. Note that by definition $A \in \tau(\mathcal{A})$ if and only if there is $G \subseteq \mathcal{B}$ such that $A = \bigcup_{B \in G} B$. We can now check that $\tau'(\mathcal{A})$ is stable under finite intersections: if we are given $G, G' \subseteq \mathcal{B}$, then

$$\bigcup_{B \in G} B \cap \bigcup_{B' \in G'} B' = \bigcup_{(B,B') \in G \times G'} B \cap B'$$

which belongs to $\tau'(\mathcal{A})$ since $\mathcal{B}$ is stable under finite intersections and $\tau'(\mathcal{A})$ is defined as the set of unions of elements of $\mathcal{B}$.

Finally $\tau'(A)$ is stable under arbitrary reunions because any reunion of reunions of elements of $\mathcal{B}$ is a reunion of elements of $\mathcal{B}$. So $\tau'(\mathcal{A})$ satisfies axioms (a), (b) and (c) hence $\tau'(\mathcal{A})$ is a topology containing $\mathcal{A}$. We now finish the proof by checking that $\tau'(\mathcal{A})$ is the smallest such topology.

If τ is any topology such that $\mathcal{A} \subseteq \tau$, then τ is stable under finite intersections and thus must contain $\mathcal{B}$. But then its stability under arbitrary reunions entails that it contains $\tau'(\mathcal{A})$ which is thus the smallest topology containing $\mathcal{A}$. By the definition of $\tau(\mathcal{A})$, we conclude $\tau(\mathcal{A}) = \tau'(\mathcal{A})$ as wanted. □

The following definition is not really needed now but it explains why we use the term subbasis.

Definition 1.70. Let (X, τ) be a topological space, let $\mathcal{B} \subseteq \tau$. We say that $\mathcal{B}$ is a **basis** for τ if every τ-open subset is a reunion of elements of $\mathcal{B}$.

Observe that by the above proposition if $\mathcal{A}$ is a subbasis of a topology τ then the set $\mathcal{B}$ of finite intersections of elements of $\mathcal{A}$ is a basis of τ. Moreover every basis is a subbasis.

Let us give an easy example: the disjoint union topology.

Exercise 1.19. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Show that the disjoint union topology is the topology generated by $\tau_X \cup \tau_Y$. Show that moreover $\tau_X \cup \tau_Y$ is a basis for the disjoint union topology.

Let us finally note the following concrete way of checking that $\mathcal{B}$ is a basis for τ.

Proposition 1.71. *Let (X,τ) be a topological space, let $\mathcal{B} \subseteq \tau$. Then $\mathcal{B}$ is a basis of τ if and only if for every open subset U and every $x \in U$ there is $B \in \mathcal{B}$ containing x such that $B \subseteq U$.*

Proof. If $\mathcal{B}$ is a base then U is a reunion of elements of $\mathcal{B}$ so in particular all the elements of U belong to some $B \in \mathcal{B}$ contained in U. For the converse note that if every $x \in U$ belongs to some $B \in \mathcal{B}$ contained in U then U must be equal to the reunion of the elements of $\mathcal{B}$ it contains. □

Corollary 1.72. *Let (X,τ) be a topological space, let $\mathcal{A} \subseteq \tau$. Then $\mathcal{A}$ is a subbasis for τ if and only if for every τ-open set U and every $x \in U$ there are $A_1, ..., A_n \in \mathcal{A}$ such that $x \in A_1 \cap \cdots \cap A_n \subseteq U$.*

Proof. This follows readily from the previous proposition along with the fact that Prop. 1.69 may now be reformulated as: a basis for $\tau(\mathcal{A})$ is the set of finite intersections of elements of $\mathcal{A}$. □

We finally explain how to make one application continuous when its range is already equipped with a topology.

Definition 1.73. Let X be a set, let (Y,τ) be a topological space and let $f : X \to Y$ be a map. The **pullback topology** $f_*\tau$ is the set

$$f_*\tau = \{f^{-1}(U) : U \in \tau\}$$

The pullback topology $f_*\tau$ is a topology because $f^{-1}(Y) = X$, $f^{-1}(\emptyset) = \emptyset$ so $f_*\tau$ contains $\emptyset$ and X and f^{-1} is compatible with all set-theoretic operations so $f_*\tau$ is stable under finite intersections and arbitrary reunions. Moreover by the characterization of continuity (Prop. 1.34) any topology making f continuous should contain $f_*\tau$ so it is the smallest topology which makes f continuous. More generally, we have the following proposition.

Proposition 1.74. *Let X be a set, let $(Y_i,\tau_i)_{i\in I}$ be a family of topological spaces and for each $i \in I$ let $f_i : X \to Y_i$ be a map. Then there is a smallest topology τ on X making f_i continuous for every $i \in I$, namely the topology generated by the reunion $\bigcup_{i\in I} f_{i*}\tau_i$.*

Moreover, the map

$$\begin{aligned} \Phi : (X,\tau) &\to \prod_{i\in I}(Y_i,\tau_i) \\ x &\mapsto (f_i(x))_{i\in I} \end{aligned}$$

is a homeomorhism onto its image.

Proof. The fact that the topology τ generated by $\bigcup_{i\in I} f_{i*}\tau_i$ is the smallest topology making each f_i continuous isa direct consequence of the fact that the topology generated by the reunion $\bigcup_{i\in I} f_{i*}\tau_i$ is the smallest topology containing each $f_{i*}\tau_i$, and for each $i \in I$ the topology $f_{i*}\tau_i$ is the smallest topology making f_i continuous.

For the second part, observe that for each $i \in I$ the injective map Φ sends every $U \in f_{i*}\tau_i$ to $\Phi(X) \cap \pi_i^{-1}(U)$ where π_i is the projection onto Y_i. Since $\bigcup_{i\in I} f_{i*}\tau_i$ generates τ_∞ while $\{\Phi(X) \cap \pi_i^{-1}(U) : U \in \tau_i, i \in I\}$ generates the topology induced by the product topology on $\Phi(X)$, this shows Φ is a homeomorphism onto its image. □

We finally have the following important way of understanding the product topology.

Corollary 1.75. *Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces. Then the product topology on $\prod_{i\in I} X_i$ is the smallest topology which makes for every $i_0 \in I$ the projection $\pi_{i_0} : \prod_{i\in I} X_i \to X_{i_0}$ continuous.*

Proof. Apply the previous proposition to $X = \prod_{i\in I} X_i$ and observe that the map Φ from the proposition is then the identity map on X ! □

1.5 Complete metric spaces

We will now focus on completeness, a feature of metric spaces which cannot be recast in purely topological terms but which will have a very important topological consequence, namely the Baire category theorem (see Section 2.8).

1.5.1 Cauchy sequences and completeness

Let (X, d) be a metric space. A sequence $(x_n) \in X^{\mathbb{N}}$ is d-Cauchy or simply Cauchy if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geqslant M$ one has $d(x_n, x_m) < \epsilon$.

Lemma 1.76. *Let (X, d) be a metric space, then every convergent sequence of elements of X is d-Cauchy.*

Proof. If $x_n \to x$ and $\epsilon > 0$, we find N such that $d(x_n, x) < \epsilon/2$ for all $n \geqslant N$. We deduce that for all $n, m \geqslant N$ we have by the triangle inequality $d(x_n, x_m) < d(x_n, x)+d(x, x_m) < \epsilon$ so that (x_n) is a Cauchy sequence. □

Definition 1.77. A metric space (X, d) is **complete** if every Cauchy sequence is convergent.

The first non-trivial example of complete metric space is provided by the reals with their usual metric. Other important examples are **Banach spaces**, i.e. normed vector spaces whose associated metric is complete. The following proposition is an easy consequence of the sequential characterization of closedness.

Proposition 1.78. *Let (X, d) be a complete metric space and let $Y \subseteq X$. Then Y is complete for the induced metric if and only if Y is closed in X.*

Proof. Let $d_Y = d_{\restriction Y\times Y}$ be the induced metric on Y.

Suppose Y is closed in X, then every d_Y-Cauchy sequence is d-Cauchy and hence has a limit in X since d is complete. Since Y is closed this limit belongs to Y, and we conclude that (Y, d_Y) is complete.

Conversely suppose Y is not closed in X, then we find a sequence (y_n) of elements of Y converging to some $x \in X \setminus Y$. By Lem. 1.76 we conclude that (y_n) is d-Cauchy, so by definition (y_n) is d_Y-Cauchy. Since $x \notin Y$, by uniqueness of the limit (y_n) is not convergent in Y so that (Y, d_Y) is not complete. □

Let us end this section with an important construction of complete metric spaces.

Definition 1.79. Let X be a set, let (Y, d) be a metric space. Denote by $\ell^\infty_d(X, Y)$ the set of functions $f : X \to Y$ such that $f(X)$ has finite diameter (such functions are also called d-bounded functions).

Observe that when the metric d is already bounded, $\ell_d^\infty(X,Y)$ is just the set of all functions $X \to Y$. The space $\ell_d^\infty(X,Y)$ is equipped with the metric d^∞ of **uniform convergence** defined by

$$d^\infty(f,g) = \sup_{x\in X} d(f(x),g(x)).$$

Exercise 1.20. Check that d^∞ is indeed well-defined and a metric (*hint :* to see that it is well-defined, observe that if we fix $x_0 \in X$ then by the triangle inequality for all $x \in X$ we have $d(f(x),g(x)) \leqslant \operatorname{diam}_d f(X) + d(f(x_0),g(x_0)) + \operatorname{diam}_d g(X)$).

Proposition 1.80. *Let X be a set and (Y,d) be a complete metric space. The metric d^∞ on $\ell_d^\infty(X,Y)$ is complete.*

Proof. Let (f_n) be a Cauchy sequence for d_∞. By definition of the metric for each $x \in X$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy and thus admits a (unique) limit which we denote by $f(x)$. Let us show that $d_\infty(f_n,f) \to 0$. Let $\epsilon > 0$, consider $N \in \mathbb{N}$ such that for all $n,m \geqslant N$ we have $d_\infty(f_n,f_m) < \epsilon$. Let $n \geqslant N$ and let $x \in X$. Then for all $m \geqslant N$ we must have $d(f_n(x),f_m(x)) < \epsilon$ so by letting m tend to $+\infty$ we have $d(f_n(x),f(x)) \leqslant \epsilon$. This shows that $d^\infty(f_n,f) \leqslant \epsilon$ and thus $f_n \to f$ as wanted: the metric d_∞ is complete. □

Example 1.81. Let X be a set, let us denote by $\ell^\infty(X,\mathbb{R})$ the set of bounded functions $f : X \to \mathbb{R}$. It is a normed vector space for the norm $\|f\|_\infty = \sup_{x\in X}|f(x)|$. The previous proposition yields that the metric associated to this norm is complete: $\ell^\infty(X,\mathbb{R})$ is a Banach space.

1.5.2 Closed subsets of vanishing diameter

We will now provide a very important consequence of completeness which is actually equivalent to it. We need the notion of *diameter*.

Definition 1.82. Let (X,d) be a metric space and let A be a non-empty subset of X. The (d-)**diameter** of A is

$$\operatorname{diam}_d(A) := \sup_{x,y\in A} d(x,y).$$

When the metric d is clear from the context, we will also write $\operatorname{diam}(A)$ for the diameter of A. A subset is called **bounded** if it is empty or has finite diameter.

Finally, a sequence (A_n) of non-empty subsets of X has **vanishing diameter** if $\operatorname{diam}(A_n) \to 0$.

Exercise 1.21. Let (X,d) be a metric space and $A \subseteq X$. Show that $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.

Remark 1.83. It follows from the previous exercise that a sequence of subsets (A_n) has vanishing diameter if and only if the sequence of closures $(\overline{A_n})$ has.

The following theorem will be used very often in these notes. The fundamental implication for us is that in a complete metric space, the intersection of every decreasing sequence of nonempty closed sets of vanishing diameter is a singleton ((i)⇒(iii)). The fact that this characterizes complete metric spaces is of theoretical importance but in these notes we will only use it when we characterize compactness in metric spaces [?].

Theorem 1.84. *Let (X,d) be a metric space. The following are equivalent:*

(i) (X,d) is a complete metric space.

(ii) Every decreasing sequence of nonempty closed sets of vanishing diameter has non-empty intersection;

(iii) The intersection of every decreasing sequence of nonempty closed sets of vanishing diameter is a singleton.

Proof. (i)⇒(ii): consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have $x_n \in F_n$. Let us show that (x_n) is Cauchy. Given $\epsilon > 0$, find $N \in \mathbb{N}$ such that $\operatorname{diam}(F_N) < \epsilon$. Then for all $n, m \geqslant N$ we have $\{x_n, x_m\} \subseteq F_N$ so that $d(x_n, x_m) < \epsilon$ by definition of the diameter. We conclude the sequence (x_n) is Cauchy.

By completeness we can consider the limit $x \in X$ of the sequence (x_n). Since each F_n is closed and contains all the x_m for $m \geqslant n$, we must have that x belongs to every F_n by Prop. 1.45 and thus $x \in \bigcap_n F_n$ which is thus non-empty.

(ii)⇒(iii) Let us show more generally that given any sequence (F_n) of subsets of X of vanishing diameter, the intersection $\bigcap_n F_n$ contains at most one point, from which the implication will be clear. If $x_1, x_2 \in \bigcap_n F_n$ then $d(x_1, x_2) \leqslant \operatorname{diam}_d(F_n)$ for all $n \in \mathbb{N}$ so $d(x_1, x_2) = 0$ and hence $x_1 = x_2$.

(iii)⇒(i): Let (x_n) be a Cauchy sequence, consider for every $n \in \mathbb{N}$ the closed set $F_n = \overline{\{x_m : m \geqslant n\}}$. Since (x_n) is Cauchy, the sequence of sets $(\{x_m : m \geqslant n\})$ has vanishing diameter, hence (F_n) has vanishing diameter as well. Define $x \in X$ by $\{x\} = \bigcap_n F_n$, we now show $x_n \to x$. Given $\epsilon > 0$, find N such that $\operatorname{diam}(F_n) < \epsilon$. We then have $d(x, x_n) < \epsilon$ for all $n \geqslant N$ because $x_n \in F_n$ and $x \in F_n$, so $x_n \to x$ as wanted. □

1.5.3 Completeness and extension of continuous functions

We will now use completeness so as to extend continuous functions whose range is a complete metric space. Let us start with a very special case of continuous functions, namely isometries. A map $f : (X, d_X) \to (Y, d_Y)$ is an **isometry** if for all $x, x' \in X$ we have $d_Y(f(x), f(x')) = d_X(x, x')$. Isometries are continuous because they are 1-Lipschitz.

Theorem 1.85. *Let (X, d_X) and (Y, d_Y) be complete metric space, let A be a dense subset of X, and let $f : A \to Y$ be an isometry with dense image. Then f extends uniquely to a surjective isometry $\tilde{f} : X \to Y$.*

Proof. Let $x \in X$, since A is dense in X we find a sequence (a_n) of elements of A such that $a_n \to a$. The sequence $(a_n)_{n\in\mathbb{N}}$ is thus Cauchy (Lem. 1.76) and since f is an isometry the sequence $(f(a_n))_{n\in\mathbb{N}}$ also is. It thus has a limit which we define to be equal to $\tilde{f}(x)$. Let us verify this limit does not depend on the choice of the sequence $a_n \to x$. If $b_n \to x$, then $d_X(a_n, b_n) \leqslant d_X(a_n, x) + d_X(x, b_n)$ so $d_X(a_n, b_n) \to 0$ and hence $d_Y(f(a_n), f(b_n)) \to 0$ so $f(a_n)$ and $f(b_n)$ must converge to the same limit. So $\tilde{f}$ is well-defined.

Moreover $\tilde{f}$ extends f because if $x \in A$ and $a_n \to x$ then $f(x) = \lim_{n\to+\infty} f(a_n)$ by continuity. Let us show that f is an isometry: let $x, y \in X$, let $a_n \to x$ and $b_n \to y$. Then

$$\begin{aligned} d_Y(\tilde{f}(x), \tilde{f}(y)) &= \lim_{n\to+\infty} d_Y(f(a_n), f(b_n)) \\ &= \lim_{n\to+\infty} d_X(a_n, b_n) \\ d_Y(\tilde{f}(x), \tilde{f}(y)) &= d_X(x, y) \end{aligned}$$

so $\tilde{f}$ is indeed an isometry. Since isometries are continuous, any isometric extension of f is determined by its restriction to A (Prop. 1.38) which if f. So $\tilde{f}$ is the unique isometry

$X \to Y$ extending f. Finally observe that $\tilde{f}(X)$ is a complete metric space for $\tilde{f}$ is an isometry and (X, d_X) is complete, so $\tilde{f}(X)$ is closed and since it contains the dense set $f(X)$, we conclude that $\tilde{f}(X) = Y$, i.e. $\tilde{f}$ is surjective. □

We will now identify a more general condition which forces a map f to send a Cauchy sequence to a Cauchy sequence.

Definition 1.86. Let (X, τ_X) be a topological space and (Y, d_Y) be a metric space, and let $f : A \subseteq X \to Y$. Given $x \in \overline{A}$, the **oscillation** of f at x, denoted by $\operatorname{osc}_f(x)$, is the quantity

$$\operatorname{osc}_f(x) = \inf_{V \in \mathcal{V}_x} \operatorname{diam}(f(V \cap A)),$$

where $\mathcal{V}_x$ denotes the neighborhood filter of x.

Observe that if $\mathcal{W}_x$ is a neighborhood basis of x, then we can compute the oscillation of f at x by only considering elements of $\mathcal{W}_x$, i.e. $\operatorname{osc}_f(x) = \inf_{V \in \mathcal{W}_x} \operatorname{diam}(f(V \cap A))$. In particular when X is metrizable and d is a compatible metric on (X, τ_X), we have

$$\operatorname{osc}_f(x) = \inf_{n \in \mathbb{N}} \operatorname{diam}(f(B_d(x, 1/n) \cap A)).$$

Moreover since $\operatorname{diam}(f(B_d(x, \epsilon) \cap A)$ is an increasing function of ϵ we actually have

$$\operatorname{osc}_f(x) = \lim_{n \to +\infty} \operatorname{diam}\left(f\left(B_d\left(x, 1/n\right) \cap A\right)\right).$$

Also note that in the above definition we always require $x \in \overline{A}$ so that $f(V \cap A)$ is nonempty, but we allow $x \notin A$.

Exercise 1.22. Let (X, d_X) and (Y, d_Y), let $f : A \subseteq X \to Y$ and let $x \in A$.

1. Show that f is continuous at x if and only if $\operatorname{osc}_f(x) = 0$.

2. Prove that if f is continuous at x then

$$\{f(x)\} = \bigcap_{\epsilon > 0} f(B(x, \epsilon)) = \bigcap_{\epsilon > 0} \overline{f(B(x, \epsilon))} = \bigcap_{n \in \mathbb{N}} \overline{f(B(x, 1/n))}.$$

The last equality from the previous exercise is the key to extending a continuous function to the set of $x \in \overline{A}$ satisfying $\operatorname{osc}_f(x) = 0$.

Proposition 1.87. *Let (X, d_X) and (Y, d_Y) be metric spaces with d_Y complete, and let $f : A \subseteq X \to Y$ be a continuous function. Then f extends uniquely to a continuous function $\tilde{f}$ on*

$$\tilde{A} := \{x \in \overline{A} : \operatorname{osc}_f(x) = 0\}.$$

Proof. Uniqueness is again a consequence of the fact that a continuous function is determined by its restriction to a dense subset (Prop. 1.38), noting that A is dense in $\tilde{A}$.

For existence, let us first define $\tilde{f}$ and then check that $\tilde{f}$ extends f and is indeed continuous.

Let $x \in \tilde{A}$, by definition $\operatorname{osc}_f(x) = 0$. Observe that the family of nonempty sets $(f(B(x, 1/n) \cap A))_{n \in \mathbb{N}}$ has vanishing diameter, hence the family of nonempty closed sets

$(\overline{f(B(x,1/n)\cap A)})_{n\in\mathbb{N}}$ has vanishing diameter. Its intersection is thus a singleton by Theorem 1.84 so we define $\tilde{f}(x)$ by

$$\{\tilde{f}(x)\} = \bigcap_{n\in\mathbb{N}} \overline{f(B(x,1/n)\cap A)}.$$

Observe that by Exercise 1.22, $\tilde{f}$ extends f.

We finally check that $\tilde{f}$ is continuous: if $x \in \tilde{A}$ and $\epsilon > 0$, take $\delta > 0$ such that $\operatorname{diam}(f(B(x,\delta)) < \epsilon$. Let $x' \in \tilde{A}$ such that $d(x,x') < \delta$, we will show that $d(\tilde{f}(x),\tilde{f}(x')) < \epsilon$. Define $\delta' := \delta - d(x,x')$. By the triangle inequality $B(x',\delta') \subseteq B(x,\delta)$ and thus $f(B(x',\delta')) \subseteq f(B(x,\delta))$. We thus have

$$\overline{f(B(x',\delta'))} \subseteq \overline{f(B(x,\delta))}.$$

In particular $\tilde{f}(x') \in \overline{f(B(x'\delta))}$. Now by assumption the set $\overline{f(B(x,\delta))}$ has diameter less than ϵ and contains $\tilde{f}(x)$ so $d(\tilde{f}(x),\tilde{f}(x')) < \epsilon$ as wanted. We conclude that $\tilde{f}$ is indeed continuous, which ends the proof. □

We used the framework of decreasing closed sets of vanishing diameter in the above proof because it provides a direct way of defining a continuous function which will be used many times in this book. But the following exercise shows that the approach through Cauchy sequence also works.

Exercise 1.23. Let (X,d_X) and (Y,d_Y) be metric spaces with d_Y complete, and let $f : A \subseteq X \to Y$ be a continuous function. Let $x \in \overline{A}$ such that $\operatorname{osc}_f(x) = 0$. Show that if $a_n \to x$ with $a_n \in A$, then $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Deduce another proof of the above proposition which follows the lines of the proof of Thm. 1.85.

The previous proposition applies particularly well when dealing with *uniformly* continuous functions.

Proposition 1.88. *Let (X,d_X) and (Y,d_Y) be metric spaces, with (Y,d_Y) complete. Let $A \subseteq X$ and $f : A \to Y$, be a uniformly continuous function. Then f extends uniquely to a uniformly continuous function $\tilde{f} : \overline{A} \to Y$.*

Proof. We want to apply Prop. 1.87. To this end, it suffices to show that for all $x \in \overline{A}$, the oscillation of the continuous function f at the point x is equal to zero.

Fix $\epsilon > 0$ and apply the definition of uniform continuity to find $\delta > 0$ such that $d_Y(f(x),f(x')) < \epsilon$ whenever $d_X(x,x') < \delta$. Then the image of any ball of radius $\delta/2$ has diameter less than ϵ, in particular the oscillation at any point $x \in \overline{A}$ is not greater than ϵ. Since $\epsilon > 0$ was arbitrary, we conclude that $\operatorname{osc}_f(x) = 0$ for all $x \in \overline{A}$.

The fact that $\tilde{f}$ is uniformly continuous is left as the next exercise. □

Exercise 1.24. Finish the proof of the above proposition: show that $\tilde{f}$ is uniformly continuous. (Hint: do a uniform version of the end of the proof of Prop. 1.87).

1.5.4 The completion of a metric space

We now show that every metric space can be turned into a complete metric space in a canonical way.

Theorem 1.89. *Let (X, d) be a metric space. There is a complete metric space $(\overline{X}, \overline{d})$ such that X is dense in $\overline{X}$ and the metric $\overline{d}$ extends d. This metric space is moreover unique in the sense that if (Y, d_Y) is another such space, then the identity map on X extends to an isometry between $(\overline{X}, \overline{d})$ and (Y, d_Y).*

Proof. Uniqueness is a direct consequence of Thm. 1.85.

For existence, observe that it suffices to find an isometry $\rho : (X, d) \to (Z, d_Z)$ where (Z, d_Z) is a complete metric space: we can then identify (X, d) to its image $\rho(X)$ in (Z, d_Z) and let $\overline{X}$ be the closure of $\rho(X)$ in Z, so that $(\overline{X}, d_Z)$ is complete by Prop. 1.78. □

Given a metric space (X, d), a complete metric space $(\overline{X}, \overline{d})$ such that X is dense in $\overline{X}$ and the metric $\overline{d}$ extends d is called a **completion** of X.

1.6 Countability and topological spaces

We will now work out some "smallness criterions" for topological spaces, all of which will be met by Polish spaces and rely on countability.

1.6.1 Separability

Definition 1.90. Let X be a topological space. We say X is **separable** if there is $A \subseteq X$ countable such that $\overline{A} = X$.

A subset A such that $\overline{A} = X$ is called a **dense** subset of X. So separable spaces are spaces admitting a countable dense subset.

Example 1.91. $\mathbb{R}$ is a separable topological space since the countable subset $\mathbb{Q}$ is dense in $\mathbb{R}$. The normed vector space $\ell^\infty(X)$ of bounded functions $X \to \mathbb{R}$ is not separable as soon as X is infinite (see Exercise **??**).

Proposition 1.92. *Any countable product of separable topological spaces is separable for the product topology.*

Proof. Let us first deal with the countable infinite case. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-empty separable spaces, and for every $n \in \mathbb{N}$ let $A_n \subseteq X_n$ be a dense subset. Let us fix a sequence $(x_n)_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X_n$. For each $n \in \mathbb{N}$ consider the set

$$B_n := A_0 \times A_1 \times \cdots \times A_n \times \{x_{n+1}\} \times \{x_{n+2}\} \times \cdots .$$

Each B_n is countable so their union $B := \bigcup_{n\in\mathbb{N}} B_n$ also is.

It remains to see why B is dense in $\prod_{n\in\mathbb{N}} X_n$. Let $(y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ and let V be a neighborhood of $(y_n)_{n\in\mathbb{N}}$. By definition of the product topology, we find $K \subseteq \mathbb{N}$ finite and for each $n \in K$ a neighborhood V_n of y_n such that V contains the set of sequences (z_n) such that for all $n \in K$ we have $z_n \in V_n$.

For each $n \in K$ there is $a_n \in A_n$ such that $a_n \in V_n$ since A_n is dense in X_n. Consider the sequence (z_n) defined by

$$z_n = \begin{cases} a_n & \text{if } n \in K \\ x_n & \text{if } n \notin K \end{cases}$$

Observe that $(z_n) \in V \cap B_N$ where $N = \max K$, witnessing that $V \cap B \neq \emptyset$ as wanted.

Now if we are dealing with a finite family of separable topological spaces $(X_i)_{i=1}^n$, we let for each i let $A_i \subseteq X_i$ be countable dense. Then the same argument as above shows that the countable set $A_1 \times \cdots \times A_n$ is dense in $\prod_{i=1}^n X_i$ which is thus separable. □

It is not true in general that a subspace of a separable space is separable (see Exercise 1.43). However, this is true in metrizable spaces, and to see this we need the notion of second-countability.

1.6.2 Second-countablity

Definition 1.93. Let X be a topological space. A family $\mathcal{B}$ of open subsets of X is called a **basis for the topology** if every open set can be written as a union of elements of $\mathcal{B}$. The topological space X is **second-countable** if its topology admits a countable basis.

The main feature of second-countablity for us will be that it is inherited by subspaces.

Proposition 1.94. *Let X be a second-countable topological space, and let $Y \subseteq X$. Then Y is second-countable for the induced topology.*

Proof. If $(U_n)_{n\in\mathbb{N}}$ is a countable basis for the topology of X, then it is straightforward to check that $(U_n \cap Y)_{n\in\mathbb{N}}$ is a countable basis for the topology of Y which is thus second-countable. □

We now turn to the relationship between second-countability and separability.

Lemma 1.95. *Every second-countable topological space is separable.*

Proof. Let X be a second-countable topological space. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis its topology. For every $n \in \mathbb{N}$ such that $U_n \neq \emptyset$, pick $x_n \in U_n$. Then the countable set $\{x_n : n \in \mathbb{N} \text{ and } U_n \neq \emptyset\}$ is dense in X: if U is a non-empty open subset of X, there is $n \in \mathbb{N}$ such that $\emptyset \neq U_n \subseteq U$ so that U contains x_n. We conclude that X is separable. □

Theorem 1.96. *Let X be a metrizable topological space. Then X is separable if and only if X is second-countable.*

Proof. We just proved that in any topological space, second-countability implies separability.

Conversely, suppose that X is a separable metrizable space. Let d be a compatible metric on X, let $A \subseteq X$ be a countable dense subset. We will show that the countable family of open balls

$$\mathcal{B} = \{B(a, r) : a \in A, r \in \mathbb{Q} \cap]0, +\infty[\}$$

is a basis for the topology, witnessing that X is second-countable. To see this, it suffices to show that every $x \in U$ belongs to some $B \in \mathcal{B}$ with $B \subseteq U$ (to check that this condition implies $\mathcal{B}$ is a basis, observe that it implies U is equal the reunion of all $B \in \mathcal{B}$ such that $B \subseteq U$).

So let $x \in U$, since U is open we find $r > 0$ such that $B(x, r) \subseteq U$. Up to taking a smaller $r > 0$ we may assume $r \in \mathbb{Q}$. Since A is dense we then find $a \in A$ which is $r/2$-close to x. The triangle inequality yields that $B(a, r/2) \subseteq B(x, r) \subseteq U$, and by symmetry we have $x \in B(a, r/2) \subseteq U$. Since $B(a, r/2) \in \mathcal{B}$, this shows that $\mathcal{B}$ is a basis for the topology. □

Corollary 1.97. *Every subspace of a separable metrizable space is also separable.*

Proof. Let X be a separable metrizable space, then X is second-countable by the previous theorem. Every subspace $Y \subseteq X$ is also second-countable as a consequence of Proposition 1.94, and hence separable by Lemma 1.95. □

Remark 1.98. The following application of second-countability will also be useful: when Y is a second-countable topological space with basis $(U_n)_{n\in\mathbb{N}}$, a map $f : X \to Y$ is continuous as soon as $f^{-1}(U_n)$ is open for every $n \in \mathbb{N}$ (by Prop. 1.68).

1.6.3 Lindelöf's lemma

Definition 1.99. Given a set X, a **cover** of X is a family of sets $(A_i)_{i\in I}$ such that

$$X \subseteq \bigcup_{i\in I} A_i.$$

A **subcover** of a cover $(A_i)_{i\in I}$ is a subfamily $(A_i)_{i\in J}$ which is still a cover of X.

When (X, τ) is a topological space, an **open cover** of X is a cover of X made of open subsets of X.

Here are some examples of open covers and subcovers :

- The families $(]-\infty, 0[,]-1, +\infty[)$ and $(]n, n+2[)_{n\in\mathbb{Z}}$ are open covers of $\mathbb{R}$.
- Any basis of open sets of a topological space forms a cover. In particular, second-countable topological spaces admit countable covers and a metric space admits the family of all its balls as an open covering.
- Given a separable metric space (X, d) and a countable dense subset Y of X, we showed that the set of open balls centered at elements of Y with rational radius actually forms a countable basis of the topology (see the proof of Thm. 1.96) and hence a countable subcover of the cover by all open balls.

As we will see now, covers of second-countable topological spaces *always* admit countable subcovers.

Lemma 1.100 (Lindelöf's lemma)**.** *Let X be a second-countable topological space. Then every open cover of X contains a countable subcover: whener $(U_i)_{i\in I}$ is an open cover of X there exists $J \subseteq I$ countable such that $X \subseteq \bigcup_{i\in J} U_i$.*

Proof. Let $(U_i)_{i\in I}$ be an open cover of X, and let $\mathcal{B} = \{V_n : n \in \mathbb{N}\}$ be a countable basis for the topology. Let $A = \{n \in \mathbb{N} : \exists i \in I, V_n \subseteq U_i\}$. For each $n \in A$, pick $i_n \in I$ such that $V_n \subseteq U_{i_n}$.

Then $(U_{i_n} : n \in A)$ is a countable subfamily of $(U_i)_{i\in I}$. To end the proof we need to show that it is also a cover of X. Let $x \in X$, we find $i \in I$ such that $x \in U_i$. Since $\mathcal{B}$ is a basis, there is $n \in \mathbb{N}$ such that $x \in V_n \subseteq U$. By definition we have $n \in A$ so that $V_n \subseteq U_{i_n}$. Since $x \in V_n$ we conclude that $x \in U_{i_n}$ so that $X = \bigcup_{n\in\mathbb{N}} U_{i_n}$ as wanted. □

Corollary 1.101. *Let X be a second-countable topological space, let $(U_i)_{i\in I}$ be a family of open subsets of X. Then $(U_i)_{i\in I}$ contains a countable subfamily with the same reunion: there is $J \subseteq I$ countable such that*

$$\bigcup_{i\in J} U_i = \bigcup_{i\in I} U_i.$$

Proof. By Prop. 1.94 the subspace $\bigcup_{i\in I} U_i$ is second-countable. Applying the previous lemma to the cover $(U_i)_{i\in I}$, we get the desired countable subfamily $(U_i)_{i\in J}$. □

1.7 Compactness

1.7.1 Definition in terms of covers

Compactness admits many equivalent formulations, and one of the most elegant of them can be seen as a strengthening of the conclusion of Lindelöf's lemma.

Definition 1.102. A topological space (X, τ) is **compact** if all its open covers admit a finite subcover: whener $(U_i)_{i\in I}$ is an open cover of X there exists $F \subseteq I$ finite such that $X \subseteq \bigcup_{i\in F} U_i$.

Exercise 1.25.

1. Show that finite sets are compact for the discrete topology
2. Show that $\mathbb{R}$ is not compact by giving an example of a countable open cover of $\mathbb{R}$ which does not admit a finite subcover.
3. Show that compactness can be checked by considering covers consisting only of elements of a given basis of the topology.

Remark 1.103. Compactness can actually be checked by considering covers consisting only of elements of a subbabis of the topology (see Exercise 1.44).

Let us give three important examples of compact spaces; the proofs that they are indeed compact will be given later using other characterizations of compactness in metrizable spaces and the fact that products of compact spaces are compact.

- Given two reals $a < b$, the interval $[a, b]$ is compact.
- The product space $\{0, 1\}^{\mathbb{N}}$ is compact. It is called the Cantor space and will be of great importance to us.
- The product space $[0, 1]^{\mathbb{N}}$ is compact. We will see later that it contains every compact metrizable space as a closed subspace.

Given a subset Y of a topological space X, when we say that Y is compact we always mean that it is compact for the induced topology. The following exercise shows that the notion of compact subset can actually be defined purely in terms of the ambient topology.

Exercise 1.26. Let X be a topological space and let $Y \subseteq X$. Show that Y is compact if and only if every cover of Y by open subsets of X admits a finite subcover.

Proposition 1.104. *The continuous image of a compact set is compact: if X is a compact topological space, Y is a topological space and $f : X \to Y$ is continuous then $f(X)$ is compact.*

Proof. Let $(V_i)_{i\in I}$ be an open cover of $f(X)$. By definition we have $f(X) \subseteq \bigcup_{i\in I} V_i$. Since $f^{-1}(f(X)) = X$, we deduce

$$X \subseteq \bigcup_{i\in I} f^{-1}(V_i).$$

By continuity each $f^{-1}(V_i)$ is open so $(f^{-1}(V_i))_{i\in I}$ is an open cover of X. It thus admits a finite subcover $(f^{-1}(V_i))_{i\in F}$. It follows that $(V_i)_{i\in F}$ is a cover of $f(X)$. We conclude that $f(X)$ is compact. □

Proposition 1.105. *Every closed subspace of a compact topological space is compact.*

Proof. Let X be a compact topological space, let F be a closed subset of X. By Exercise 1.26 we need to show that every open cover F by open subsets of X admits a finite subcover. Let $(U_i)_{i\in I}$ be such a cover, let $i_0 \notin I$ and consider the open set $U_{i_0} = X \setminus F$. Then $(U_i)_{i\in I\cup\{i_0\}}$ is an open cover of X which thus admits a finite subcover $(U_i)_{i\in F}$ with $F \subseteq I \cup \{i_0\}$ finite. By the definition of U_{i_0} we conclude that $(U_i)_{i\in F\setminus\{i_0\}}$ is a finite subcover of F. □

We will end this section with a very useful reformulation of compactness which is basically obtained by taking complements.

Definition 1.106. A family of sets $(A_i)_{i\in I}$ has the finite intersection property if for every $F \subseteq I$ finite, one has $\bigcap_{i\in F} A_i \neq \emptyset$.

Note that if $\bigcap_{i\in I} A_i \neq \emptyset$ then the family $(A_i)_{i\in I}$ has the finite intersection property.

Proposition 1.107. *Let (X, τ) be a topological space. Then X is compact if and only if every family of closed sets $(F_i)_{i\in I}$ which has the finite intersection property actually has non-empty intersection: $\bigcap_{i\in I} F_i \neq \emptyset$.*

Proof. If X is compact, suppose by contradiction $(F_i)_{i\in I}$ is a family of closed subsets with the intersection property but with trivial intersection. Then $(X \setminus F_i)_{i\in I}$ is an open cover of X with no finite subcover, a contradiction. So $\bigcap_{i\in I} F_i \neq \emptyset$ and we conclude that every family of closed sets with the finite intersection property has non-empty intersection.

Conversely, suppose X is not compact. We then find an open cover $(U_i)_{i\in I}$ of X with no finite subcover. We thus have a family of closed subsets $(X \setminus U_i)_{i\in I}$ which has the finite intersection property but whose intersection is empty. □

1.7.2 Compactness in metric spaces

1.7.2.1 Precompactness

Applying the definition of compactness to the cover of a metric space by open balls of a fixed radius leads to the notion of precompactness.

Definition 1.108. A metric space (X, d) is **precompact** if for every $\epsilon > 0$ there is $F \subseteq X$ finite such that

$$(B(x, \epsilon))_{x\in F}$$

is a cover of X.

Proposition 1.109. *Every compact metric space is precompact.*

Proof. This is a straightforward consequence of the definition of compactness applied to the covers $(B(x, \epsilon))_{x\in X}$ for every $\epsilon > 0$. □

Here is a simple reformulation of the notion of precompactness for a metric space (X, d): a subset $A \subseteq X$ is **ϵ-dense** in X if for all $x \in X$ there is $a \in A$ such that $d(x, a) < \epsilon$. Now the precompactness of (X, d) is equivalent to the following statement: for every $\epsilon > 0$, there is a finite subset of X which is ϵ-dense in X.

Proposition 1.110. *Every precompact metric space is bounded (i.e. has finite diameter).*

Proof. Let $F \subseteq X$ be a finite subset which is 1-dense. Let us show that the diameter of X is at most $\operatorname{diam}(F)+2$: given $x, x' \in X$ we find $f, f' \in F$ such that $d(x,f) < 1$ and $d(x', f') < 1$ by 1-density . Then $d(f,f') \leqslant \operatorname{diam}(F)$ so by the triangle inequality

$$d(x,x') \leqslant d(x,f) + d(f,f') + d(f',x') \leqslant \operatorname{diam}(F) + 2.$$

Since F is finite it has finite diameter, so we can conclude from the previous inequation that $\operatorname{diam}(X)$ is finite. □

Corollary 1.111. *Every compact metric space is bounded.*

Proof. By Proposition 1.109 every compact space is precompact, hence bounded by the previous proposition. □

We now give a sequential characterization of precompactness which will allow us to recover the well-known characterization of compactness in terms of subsequences (a sequence (v_n) is a **subsequence** of a sequence (u_n) if there is an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $v_n = u_{\varphi(n)}$).

Proposition 1.112. *Let (X, d) be a metric space. The following are equivalent*

(i) (X, d) is precompact;

(ii) every sequence has a Cauchy subsequence.

Proof. (i)⇒(ii): suppose (X,d) is precompact, let (x_n) be a sequence of elements of X. We first build by induction a decreasing family (A_n) of infinite subsets of $\mathbb{N}$ such that $A_0 = \mathbb{N}$ and for every $n > 0$ we have $\operatorname{diam}(\{x_m : m \in A_n\}) \leqslant \frac{1}{n}$.

We start with A_0, and then assuming A_n has been constructed, we use precompactness to cover X by finitely many balls $(B_i)_{i=1}^k$ of radius $\frac{1}{2(n+1)}$. Since A_n is infinite, there is $i \in \{1, ..., k\}$ such that for infinitely many $m \in A_n$, we have $x_m \in B_i$. We then let $A_{n+1} = \{m \in A_n : x_m \in B_i\}$ and observe that $\operatorname{diam}(\{x_m : m \in A_{n+1}\}) \leqslant \operatorname{diam} B_i \leqslant \frac{1}{n+1}$ as wanted. This completes the construction. Now let $\varphi(n) = \min\{m : m \in A_n\}$, then $(x_{\varphi(n)})$ is the desired Cauchy subsequence.

(ii)⇒(i): by contrapositive, suppose (X,d) is not precompact, then there is some $\epsilon > 0$ such that no finite subset of X is ϵ-dense. This allows us to construct by induction a sequence (x_n) such that $d(x_n, x_m) \geqslant \epsilon$ for all $n \neq m$: start with an arbitrary $x_0 \in X$, and when $x_0, ..., x_n$ have been built take x_{n+1} witnessing that the finite set $\{x_0, ..., x_n\}$ is not ϵ-dense. Since the property $d(x_n, x_m) \geqslant \epsilon$ for all $n \neq m$ will be inherited by subsequences, such a sequence (x_n) cannot have a Cauchy subsequence. □

The following exercise shows that one can characterize precompact subsets of metric spaces "from the outside".

Exercise 1.27. Let (X,d) be a metric space.

1. Show that a subspace $Y \subseteq X$ is precompact for the induced metric if and only if for every $\epsilon > 0$ there is $F \subseteq X$ finite such that $Y \subseteq \bigcup_{x \in F} B_d(x, \epsilon)$

2. Deduce that $Y \subseteq X$ is precompact for the induced metric if and only if $\overline{Y}$ is.

1.7.2.2 Mesh property

We say that a metric space (X, d) has the **mesh property** if for every open cover of X, there is $\epsilon > 0$ such that every open ball of radius ϵ is contained in some U_i: for every $x \in X$, there is $i \in I$ such that

$$B(x, \epsilon) \subseteq U_i.$$

Such an ϵ is called a **mesh** of the cover $(U_i)_{i \in I}$. We will now reuse the ideas from Lindelöf's lemma in our metric context to show that compact metric spaces have the mesh property.

Lemma 1.113. *Let (X, d) be a compact metric space, then (X, d) has the mesh property.*

Proof. Define a set $A := \{(x, \epsilon) \in X \times \mathbb{R}_+^* : \exists i \in I, B(x, 2\epsilon) \subseteq U_i\}$. Then since $(U_i)_{i \in I}$ is an open cover of X and open balls form a basis of the topology, we have that

$$(B(x, \epsilon))_{(x,\epsilon) \in A}$$

is an open cover of X. By compactness, we find $n \in \mathbb{N}$, elements $x_1, ..., x_n \in X$, radiuses $\epsilon_1, ..., \epsilon_n$ and indices $i_1, ..., i_n \in I$ such that

$$X \subseteq \bigcup_{k=1}^{n} B(x_k, \epsilon_k)$$

and for every $k = 1, ..., n$ we have $B(x_i, 2\epsilon_i) \subseteq U_{i_k}$. Let $\epsilon = \min\{\epsilon_1, ..., \epsilon_n\}$, we will see that this ϵ works. Indeed if $x \in X$, then there is $k \in \{1, ..., n\}$ such that $x \in B(x_k, \epsilon_k)$, and then since $B(x_k, 2\epsilon_k) \subseteq U_{i_k}$ the triangle inequality yields $B(x, \epsilon) \subseteq U_{i_k}$ as wanted. □

Remark 1.114. See Exercise **??** for a proof of this proposition using Lipschitz functions.

We can now obtain the uniform continuity of continuous functions on compact spaces as a consequence of the existence of a mesh for an open cover.

Proposition 1.115. *Let (X, d_X) be a compact metric space, let (Y, d_Y) be a metric space and let $f : X \to Y$ be a continuous function. Then f is uniformly continuous: for every $\epsilon > 0$ there is $\delta > 0$ such that for all $x, x' \in X$ satisfying $d_X(x, x') < \delta$ we have $d_Y(f(x), f(x')) < \epsilon$.*

Proof. Let $\epsilon > 0$, and consider the cover $(f^{-1}(B_{d_Y}(f(x), \epsilon/2)))_{x \in X}$. Let δ be a mesh for this cover, we will see that this δ works.

Let $x, x' \in X$ with $d_X(x, x') < \delta$. Then there is some $x'' \in X$ such that $B_{d_X}(x, \delta) \subseteq f^{-1}(B_{d_Y}(f(x''), \epsilon/2))$. So x and x' belong to $f^{-1}(B_{d_Y}(f(x''), \epsilon/2)$, which means that $d_Y(f(x), f(x'')) < \epsilon/2$ and $d_Y(f(x'), f(x'')) < \epsilon/2$. By the triangle inequality we conclude that

$$d_Y(f(x), f(x')) \leqslant d_Y(f(x), f(x'')) + d_Y(f(x''), f(x')) < \epsilon$$

as wanted. □

1.7.2.3 Characterizing compactness in metric spaces

Theorem 1.116. *Let (X, d) be a metric space. The following are equivalent:*

(i) X is compact;

(ii) (X, d) is precompact and complete;

(iii) Every sequence of elements of X has a converging subsequence;

(iv) (X, d) is precompact and has the mesh property.

Proof. (i)⇒(ii): Suppose X is compact. Then X is precompact (Prop. 1.109). To see that (X, d) is complete, we use the characterization provided by Theorem 1.84: given a decreasing family of nonempty closed subsets (F_n) of vanishing diameter, we must show it has nonempty intersection. By compactness we only need to show that it has the finite intersection property (see Prop. 1.107). So let $n_1, ..., n_k \in \mathbb{N}$, and let $N = \max(n_1, ..., n_k)$, then since the sequence (F_n) is decreasing we have $F_N \subseteq \bigcap_{i=1}^k F_{n_i}$. The set F_N being nonempty, we conclude that $\bigcap_{i=1}^k F_{n_i}$ is nonempty. So (F_n) has the finite intersection property and hence has nonempty intersection by compactness.

(ii)⇒(iii) Let (x_n) be a sequence of elements of X, then by precompactness it has a Cauchy subsequence. By completeness, such a subsequence converges.

(iii)⇒(iv): If every sequence has a converging subsequence, then since converging sequences are Cauchy (Lem. 1.76) we conclude that every sequence has a Cauchy subsequence. So by Prop. 1.112, (X, d) is precompact.

Let us now prove by contradiction that (X, d) also satisfies the mesh property. Suppose not, let $(U_i)_{i \in I}$ be an open cover of X, then given any $\epsilon > 0$ we can find some $x \in X$ such that $B(x, \epsilon)$ is contained in no U_i. Applying this for every n with $\epsilon_n = \frac{1}{n}$ we find a sequence (x_n) such that $B(x_n, 1/n)$ is contained in no U_i. By asumption it has a subsequence $(x_{\varphi(n)})$ converging to some $x \in X$. Now let $i \in I$ such that $x \in U_i$. We find ϵ such that $B(x, 2\epsilon) \subseteq U_i$. But for large enough n, we have $d(x, x_{\varphi(n)}) < \epsilon$ and $1/\varphi(n) < \epsilon$ so that by the triangle inequality

$$B\left(x_{\varphi(n)}, \frac{1}{\varphi(n)}\right) \subseteq B(x, 2\epsilon) \subseteq U_i,$$

a contradiction.

(iv)⇒(i): Suppose (X, d) is precompact and has the mesh property, let us show X is compact. Let $(U_i)_{i \in I}$ be an open cover of X, let ϵ be a mesh for this cover. Then by precompactness we find a finite set $\{x_1, ..., x_n\}$ which is ϵ-dense in X. Since ϵ is a mesh for each $k \in \{1, ..., n\}$ there is $i_k \in I$ such that $B(x_k, \epsilon) \subseteq U_{i_k}$. We conclude that $(U_{i_k})_{k=1}^n$ is a finite subcover of X. So every open cover of X has a finite subcover, in other words X is compact. □

Note that item (3) above does not refer to the metric itself but only to the topology. We thus have the following important corollary.

Corollary 1.117. *A metrizable topological space X is compact if and only if every sequence of elements of X has a converging subsequence.*

Remark 1.118. The above theorem also shows that if X is a metrizable compact topological space, then *every* compatible metric on X is precompact and complete. In Exercise 2.3 we will actually show that a metrizable topological space is compact if and only if all its compatible metrics are complete.

Corollary 1.119. *Let (X, d) be a complete metric space. A subset $Y \subseteq X$ is precompact if and only if its closure is compact.*

Proof. Observe that if F is ϵ-dense in Y, then F is 2ϵ-dense in $\overline{Y}$ so if Y is precompact its closure $\overline{Y}$ also is. Moreover $\overline{Y}$ is a complete metric space because X is and $\overline{Y}$ is closed, so by the above theorem $\overline{Y}$ is compact.

Conversely if $\overline{Y}$ is compact then it is precompact □

1.7.3 Tychonov's theorem

On of the most fundamental properties of compactness is that it is stable under products.

Theorem 1.120 (Tychonov). *Any product of compact spaces is compact.*

In this section, we are only going to prove Tychonov's theorem in the case of a countable product of compact metric spaces. Our main reason for doing so is that Tychonov's theorem in its above form is actually equivalent to the axiom of choice which we try to avoid as much as possible **??** so as to get constructive proofs. A proof of the general version of Tychonov's theorem is also given in exercise **??**.

For this, we use the characterization of compactness in terms of subsequences. The key idea is that we can iterate taking subsequences. The following exercise is a good warmup.

Exercise 1.28. Show that any finite product of compact metrizable spaces is compact metrizable.

Theorem 1.121. *Any countable product of compact metrizable spaces is compact metrizable.*

Proof. By the previous exercise we only need to deal with the countable infinite case. So let $(X_n)_{n\in\mathbb{N}}$ be a countable family of compact metrizable spaces. For notational simplicity, we go back to the set-theoretic definition of products and view elements of the product space as maps $f : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} X_n$ such that $f(n) \in X_n$ for all n.

We have shown in Prop. 1.59 that the product $\prod_{n\in\mathbb{N}} X_n$ is metrizable, so by the characterization of compactness for metric spaces we need to show that every sequence of elements of $\prod_{n\in\mathbb{N}}(X_n)$ has a converging subsequence.

So let $(f_m)_{m\in\mathbb{N}}$ be a sequence of elements of $\prod_{n\in\mathbb{N}}(X_n)$. We will build by induction a family (φ_n) of increasing maps $\mathbb{N} \to \mathbb{N}$ and elements $x_n \in X_n$ such that for each n, the sequence

$$(f_{\varphi_0\cdots\varphi_n(m)}(n))_{m\in\mathbb{N}}$$

converges to x_n.

We start by applying the compactness of X_0 to find an increasing $\varphi_0 : \mathbb{N} \to \mathbb{N}$ and $x_0 \in X_0$ such that $f_{\varphi_0(m)}(0)$ converges to $x_0 \in X_0$. Then, $\varphi_0, ..., \varphi_n$ and $x_0, ..., x_n$ having been built, we apply the compactness of X_{n+1} to the sequence $(f_{\varphi_0\cdots\varphi_n(m)}(n+1))_{m\in\mathbb{N}}$ to find an increasing $\varphi_{n+1} : \mathbb{N} \to \mathbb{N}$ and $x_{n+1} \in X_{n+1}$ such that $f_{\varphi_0\cdots\varphi_{n+1}(m)}(n+1)$ converges to x_{n+1} as m tends to $+\infty$.

Now consider the map $\varphi : \mathbb{N} \to \mathbb{N}$ given by $\varphi(m) = \varphi_0 \cdots \varphi_m(m)$. Let us prove that $f_{\varphi(m)} \to (x_n)_{n\in\mathbb{N}}$ as m tens to ∞. By Theorem 1.116 we need to show that for all $n \in \mathbb{N}$ we have $f_{\varphi(m)}(n) \to x_n$. So fix $n \in \mathbb{N}$ and let V be a neighborhood of x_n.

By the definition of φ_n we have that $f_{\varphi_0\cdots\varphi_n(m)}(n) \to x_n$ as $m \to +\infty$. So there is $M \in \mathbb{N}$ such that for all $m \geqslant M$, we have $f_{\varphi_0\cdots\varphi_n(m)}(n) \in V$. Up to replacing M by $\max(M, n)$, we may as well assume $M > n$. Now for all $m \geqslant M$, we have $\varphi_{n+1} \cdots \varphi_m(m) \geqslant m$, and hence $f_{\varphi_0\cdots\varphi_n\varphi_{n+1}\cdots\varphi_m(m)}(n) \in V$, which by definition means $f_{\varphi(m)} \in V$ as wanted. □

1.7.4 Compactness and separation

We now will see that in compact spaces, Hausdorfness allows one not only to separate distinct points by open sets, but also disjoint compact sets. Let us first give a precise meaning to the notion of separation, which is a recurrent theme in descriptive set theory.

Definition 1.122. Let X be a set, let A and B be two disjoint subsets of X. We say that A and B are **separated** by two other subets C and D if we have $A \subseteq C$, $B \subseteq D$ and $C \cap D = \emptyset$.

Of course A and B are always separated by themselves, but the point of the above definition is to separate them by nicer sets. For instance, Hausdorfness may be reformulated as: disjoint singletons can be separated by open subsets. Here is a stronger statement.

Proposition 1.123. *Let X be a Hausdorff topological space. Then if K is a compact subset of X and $x \in X \setminus K$ then there are disjoint open subsets U and V with $K \subseteq U$ and $x \in V$. In other words, we can separate points from compact subsets by open subsets.*

Proof. Let $K \subseteq X$ be compact for the induced topology, and let $x \notin K$. Since X is Hausdorff, the family of all open subsets of X which are disjoint from some open subset containing x is a cover of $X \setminus \{x\}$, in particular it is a cover of K. By Exercise 1.26 such a cover has a finite subcover $(U_1, ..., U_n)$ and by definition for each $i \in \{1, ..., n\}$ there is an open set V_i containing x which is disjoint from U_i.

The open set $V := \bigcap_{i=1}^n V_i$ is then disjoint from the open set $U := \bigcup_{i=1}^n U_i$ which contains K. Moreover $x \in V$ so U and V are as wanted. □

In the above proof, we used an important trick which we single out.

Trick A (Separation trick)**.** *Suppose A and B are sets with A covered by $(A_i)_{i\in I}$. Suppose for each $i \in I$ we have another set B_i such that A_i and B are separated by A_i and B_i. Then A is separated from B by $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} B_i$.*

Proof. Indeed $A \subseteq \bigcup_{i\in I} A_i$ because $(A_i)_{i\in I}$ covers A and $B \subseteq \bigcap_{i\in I} B_i$ because for each $i \in I$ we have $B \subseteq B_i$. Moreover $\bigcup_{i\in I} A_i \cap \bigcap_{i\in I} B_i \subseteq \bigcup_{i\in I} A_i \cap B_i = \emptyset$. □

Exercise 1.29. Show that in a Hausdorff topological space, any two disjoint compact sets can be separated by open sets. (Hint: if K and L are disjoint compact, use the previous proposition and compactness to find a finite open cover $(U_i)_{i=1}^n$ of K and open sets $V_i \supseteq L$ disjoint from U_i. Conclude by applying the above trick).

Let us now move to an important consequence of the above separation result.

Proposition 1.124. *Let X be a Hausdorff topological space. Then every compact subset of X is closed.*

Proof. Let $K \subseteq X$ be compact, let $x \notin K$. By the previous proposition we get disjoint open sets U and V with $K \subseteq U$ and $x \in V$. In particular K is disjoint from V which is a neighborhood of x. We conclude K is closed. □

Corollary 1.125. *Let X be a compact Hausdorff topological space, let $Y \subseteq X$. Then Y is compact if and only if it is closed.*

Proof. The direct implication is the previous proposition, while the converse is Prop. 1.105. □

In the metric case, we can actually separate closed sets from compact ones by open sets using the following result.

Proposition 1.126. *Let (X, d) be a metric space. Let K be a compact subspace of X and let F be a closed subset. Then there is $\epsilon > 0$ such that for all $k \in K$ and $f \in F$, we have*

$$d(x, f) \geqslant \epsilon.$$

Proof. Suppose not. Then we can build sequences $(k_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}} \in F^{\mathbb{N}}$ such that $d(k_n, f_n) < 1/n$ for all $n \in \mathbb{N}$. Since K is compact, by taking a subsequence we may assume that $k_n \to k$ for some $k \in K$. Then since $d(k_n, f_n) \to 0$ we also have $f_n \to k$, and since F is closed we conclude $k \in F$, which contradicts the disjointness of F and K. □

Theorem 1.127. *Let X be a compact topological space, let Y be a Hausdorff topological space. Let $f : X \to Y$ be a continuous map. The following are true:*

(1) whenever $F \subseteq X$ is closed, its image $f(F)$ is compact in Y and

(2) if f is injective, it is a homeomorphism onto its image.

Proof. (1): Let $F \subseteq X$ closed, then F is compact because X is, so by Prop. 1.104 $f(F)$ is compact.

(2): The map f is continuous injective so to show that f is a homeomorphism onto its image we need to show that f maps open subsets of X to open subsets of $f(X)$, or equivalently that f maps closed subsets of X to closed subsets of $f(X)$. But if $F \subseteq X$ is closed we have just seen that $f(F)$ is compact in Y hence closed in Y because Y is Hausdorff. In particular $f(F)$ is closed in $f(X)$ and we conclude that f is a homeomorphism onto its image. □

1.7.5 Compactness in $\mathbb{R}$

Compactness is very useful in the topological space $\mathbb{R}$. Recall that $\mathbb{R}$ a metrizable topological space with a nice compatible metric provided by the *complete* metric d associated to the absolute value:

$$d(x, y) = |x - y|.$$

When we view $\mathbb{R}$ as a metric space, we will always be considering this metric. Let us first understand exactly which subsets of $\mathbb{R}$ are compact.

Proposition 1.128. *Let $a \leqslant b$ be two reals. Then the closed interval $[a, b]$ is compact.*

Proof. First, $[a, b]$ is closed because it is equal to the closed ball of radius $(a - b)/2$ around $(a + b)/2$. The compatible metric $d(x, y) = |x - y|$ thus restricts to a complete metric on $[a, b]$. Moreover, if $\epsilon > 0$, by the archimedean property of the reals there is $n \in \mathbb{N}$ such that $n\epsilon \geqslant 2(b - a)$. It is then straightforward to check that $[a, b]$ is covered by the finite family of ϵ-balls $(B(a + k\epsilon/2, \epsilon))_{k=0}^{n}$, so $[a, b]$ is precompact. We conclude that $([a, b], d)$ is a precompact complete metric space, hence $[a, b]$ is compact by Thm. 1.116. □

Theorem 1.129. *The compact subspaces of $\mathbb{R}$ are exactly the closed bounded subsets of $\mathbb{R}$.*

Proof. If $K \subseteq \mathbb{R}$ is compact, then K is bounded by Cor. 1.111 and closed by Prop. 1.124. Conversely if K is bounded and closed, let $M > 0$ be a bound for the diameter of K. Fix $x_0 \in K$, then K is contained in $[x_0 - M, x_0 + M]$ which is compact by Prop. 1.128. So K is a closed subset of a compact space hence K is compact. □

The following is a fundamental result in analysis.

Theorem 1.130. *Every non-empty compact subset of* $\mathbb{R}$ *has a minimum and a maximum.*

Proof. Let $K \subseteq \mathbb{R}$ be compact and non empty. Consider the family of non-empty closed sets $(]-\infty, x] \cap K)_{x \in X}$. This family is closed under finite intersections because

$$(]-\infty, x_1] \cap K) \cap \cdots \cap (]-\infty, x_n] \cap K) =]-\infty, \min(x_1, ..., x_n)] \cap K,$$

so since it consists of non-empty sets it has the finite intersection property. Since K is compact, Prop. 1.107 yields that the set $L := \bigcap_{x \in K}]-\infty, x] \cap K$ is nonempty. Let $x_0 \in L$, then $x_0 \in K$ and for every $x \in K$ we have $x_0 \in]-\infty, x]$ so $x_0 \leqslant x$. We conclude that x_0 is the minimum of K (note that L must then be a singleton).

For the maximum, the same proof works by considering the family $([x, +\infty[\cap K)_{x \in X}$ instead. □

1.8 Local compactness and one point compactifications

Definition 1.131. A topological space X is **locally compact** if every $x \in X$ has a neighborhood basis consisting of compact subsets.

As a first example, note that the discrete topology is always locally compact.

Example 1.132. Here is a less trivial example: $\mathbb{R}$ is locally compact. Indeed if $x \in \mathbb{R}$, closed intervals of the form $[x - \epsilon, x + \epsilon]$ for $\epsilon > 0$ form a basis of compact neighborhoods of x.

Unlike for compact spaces, arbitrary products of locally compact spaces are not locally compact. However we still have the following.

Proposition 1.133. *Any finite product of locally compact spaces is locally compact.*

Proof. It suffices to show that the product of two locally compact spaces is locally compact, so let X and Y be locally compact, let $(x, y) \in X \times Y$. Let W be a neighborhood of (x, y) then by definition of the product topology we find neighborhoods U of x and V of Y such that $U \times V \subseteq W$. Since X and Y are locally compact, we find $K \subseteq U$ compact neighborhood of x and $L \subseteq V$ compact neighborhood of y, so that $K \times L \subseteq W$ is a compact neighborhood of (x, y). □

Exercise 1.30. Show $\mathbb{R}^{\mathbb{N}}$ is not locally compact. (Hint: show by contradiction that every compact set has empty interior).

To see that a Hausdorff topological space is locally compact, it actually suffices to check that each point admits *one* compact neighborhood.

Theorem 1.134. *Let X be a Hausdorff topological space. Then the following are equivalent:*

(i) X is locally compact;

(ii) every $x \in X$ admits a compact neigbhorhood.

Proof. The implication (i)⇒(ii) follows from the fact that a basis of neighborhoods is never empty.

For (ii)⇒(i), assume every $x \in X$ admits a compact neigbhorhood. Let $x \in X$, let V be a neighborhood of x. We need to find a compact neigborhood of x inside V.

By assumption we can find a compact neighborhood K of x. Up to replacing V by $V \cap K$, we may as well assume $V \subseteq K$. Consider the compact space $K \setminus V$. There are disjoint open sets U_1 and U_2 such that $x \in U_1$ and $K \setminus V \subseteq U_2$. Since V is a neighborhood of x, we find $W \subseteq V$ open containing x. Now let us see why the neighborhood F of x defined by $F = \overline{W \cap U_1}$ is as wanted. First, F is compact because it is a closed subset of the compact set K. Moreover U_1 is disjoint from the open set U_2, its closure also is and so F is disjoint from U_2. In particular F is disjoint from $K \setminus V$, which since $F \subseteq K$ means that F is contained in V as wanted. □

Corollary 1.135. *Every compact Hausdorff topological space is locally compact.*

Proof. Let X be compact Hausdorff, then for every $x \in X$ the set X is a compact neighborhood of x. Since X is Hausdorff the preceding theorem allows us to conclude X is locally compact. □

We now relate locally compact spaces to compact spaces by "adding a point at infinity", generalizing the construction of the one point compactification of $\mathbb{N}$.

Definition 1.136. Let X be a Hausdorff locally compact space which is not compact. Its one point compactification is the set $X \sqcup \{\infty\}$ equipped with the topology whose open sets are the open subsets of X along with sets of the form $\{\infty\} \cup X \setminus K$ where K ranges over compact subsets of X.

Exercise 1.31. Check that the one point compactification of a Hausdorff locally compact space is ineed a topological space, that it is compact and that X embeds into it.

1.9 Connectedness

Ultrametric, example: $\mathbb{Q}_p$, for lc we have td implies zero dim. Intervals are connected.

1.10 Urysohn's metrization theorem

In this section we give a characterization of the second countable topological spaces which admit a compatible metric. We first give a necessary separation condition called **normality**.

Definition 1.137. A topological space X is **normal** if disjoint closed sets can be separated by open sets: for every disjoint closed subsets $F, G \subseteq X$ there are disjoint open subsets U and V such that $F \subseteq U$ and $G \subseteq V$.

Proposition 1.138. *Every metrizable topological space is normal.*

Proof. Let X be a topological space and d be a compatible metric. Let $F, G \subseteq X$ be disjoint closed sets. Since F and G are closed and disjoint, we have $d(x, G) > 0$ for all $x \in F$ and $d(x, F) > 0$ for all $x \in G$. We then define

$$U = \bigcup_{x \in F} B_d(x, d(x, G)/2) \text{ and } V = \bigcup_{x \in G} B_d(x, d(x, F)/2).$$

Being unions of open balls, both U and V are open and we clearly have $F \subseteq U$ and $G \subseteq V$. Let us finish the proof by showing U and V are disjoint. If $y \in U \cap V$ then we find $x_1 \in F$ with $d(x_1, y) < d(x_1, G)/2$ and $x_2 \in G$ with $d(x_2, y) < d(x_2, F)$. By the triangle inequality we have

$$d(x_1, x_2) \leqslant d(x_1, y) + d(x_2, y) < \frac{d(x_1, G) + d(x_2, F)}{2}.$$

But by the definition of the distance functions $d(x_1, x_2) \geqslant d(x_1, G)$ and $d(x_1, x_2) \geqslant d(x_2, F)$ so $d(x_1, x_2) \geqslant \frac{d(x_1,G)+d(x_2,G)}{2}$, a contradiction. □

Proposition 1.139. *Every compact Hausdorff topological space is normal.*

Proof. Let X be a compact Hausdorff topological space. If F and G are two disjoint closed subsets of X they are compact (Prop. 1.105), so by Exercise 1.29 they can be separated by open subsets. □

The following easy reformulation of normality will prove more useful.

Exercise 1.32. Show that a topological space is normal if and only if for every closed set F contained in an open set U, there is an open set V such that $F \subseteq V$ and $\overline{V} \subseteq U$.

Recall that the topological space $[0, 1]^{\mathbb{N}}$ is metrizable and that every subspace of a metrizable space is metrizable. We are going to prove that every a second-countable Hausdorff normal topological space X is metrizable by showing that X is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$. To this end, we need a big supply of continuous functions $X \to [0, 1]$ which are provided by the following key lemma.

Lemma 1.140. *Let X be a normal topological space and let F, G be disjoint closed subsets of X. There is a continuous function $f : X \to [0, 1]$ such that $f(F) = \{0\}$ and $f(G) = \{1\}$.*

Proof. Using the reformulation of normality provided by Exercise 1.32, we will first build by induction a family of open sets $(U_q)_{q\in\mathbb{Q}\cap]0,1]}$ containing F and contained in $X \setminus G$ such that for all $q, r \in \mathbb{Q} \cap [0, 1]$ with $q < r$, we have

$$\overline{U_q} \subseteq U_r.$$

We initiate our construction with $U_1 = X \setminus G$, and U_0 an open set such that $F \subseteq U_0$ and $\overline{U_0} \subseteq U_1$. We then let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ with $q_0 = 0$ and $q_1 = 1$.

Assuming that $U_{q_0}, ..., U_{q_n}$ have been built for some $n \geqslant 1$, let us construct $U_{q_{n+1}}$. We first find $k, l \in \{0, ..., n\}$ so that q_k is the greatest element of $\{q_0, ..., q_n\}$ less than q_{n+1} while q_l is the smallest element of $\{q_0, ..., q_n\}$ greater than q_{n+1}. We then apply normality to $\overline{U}_{q_k} \subseteq U_{q_l}$ to find $U_{q_{n+1}}$ with

$$\overline{U_{q_k}} \subseteq U_{q_{n+1}} \text{ and } \overline{U_{q_{n+1}}} \subseteq U_{q_l}.$$

By construction in the end we obtain a family of open sets $(U_q)_{q\in\mathbb{Q}\cap[0,1]}$ containing F and contained in $X \setminus G$ such that for any $q, r \in \mathbb{Q} \cap [0, 1]$ with $q < r$, we have

$$\overline{U_q} \subseteq U_r.$$

Let us now define our function $f : X \to [0,1]$ by

$$f(x) = \inf(\{1\} \cup \{q \in \mathbb{Q} \cap [0,1] : x \in U_q\})$$

Observe that by definition we have the following property: for all $x \in U_q$ we have $f(x) \leqslant q$ and for all $x \notin U_q$ we have $f(x) \geqslant q$. Since F is contained in every U_q, we then have $f(F) \subseteq \{0\}$, and since each U_q is disjoint from G we also have $f(G) \subseteq \{1\}$. We thus have to understand why f is continuous.

Let $b \in]0,1]$, we have $f^{-1}([0,b[) = \bigcup_{q<b} U_q$: the inclusion from right to left follows from the fact that for every $q \in \mathbb{Q} \cap [0,1]$ we have $f(U_q) \subseteq [0,q]$, and conversely if $f(x) < b$ we may find a rational q such that $f(x) < q < b$ and it follows that $x \in U_q$.

Now let $a \in [0,1[$, then $f^{-1}(]a,1]) = \bigcup_{q>a} X \setminus \overline{U_q}$. Indeed if $f(x) > a$ we find ϵ such that $x \notin U_r$ for all $r \in]a, a+\epsilon[\cap \mathbb{Q}$. We then pick such a r and chose some rational $q \in]a,r[$: we then have $\overline{U_q} \subseteq U_r$ and since $x \notin U_r$ in particular $x \notin \overline{U_q}$. The reverse inclusion $\bigcup_{q>a} X \setminus \overline{U_q} \subseteq f^{-1}(]a,1])$ follows immediatly from the fact that for all $x \notin U_q$ we have $f(x) \geqslant q$.

Since the topology of $[0,1]$ is generated by intervals of the form $[0,b[$ and $]a,1]$ and their preimages by f are open, we conclude that the preimage of any open subset of $[0,1]$ is open: the map f is continuous as wanted. □

Theorem 1.141. *Let X be a second-countable Hausdorff topological space. Then the following are equivalent:*

(i) X is normal;

(ii) X is metrizable;

(iii) X is homeomorphic to a subspace of the Hilbert cube $[0,1]^{\mathbb{N}}$.

Proof. The implication (ii)⇒(i) is Prop. 1.138, while (iii)⇒(ii) follows from the fact that $[0,1]^{\mathbb{N}}$ is metrizable (by Prop. 1.59) and subspaces of metrizable spaces are metrizable.

So we only need to prove (i)⇒(iii). Let X be a normal second-countable Hausdorff topological space. Let $(U_n)_{n\in\mathbb{N}}$ be a basis for its topology. We need to build enough continuous functions $f_n : X \to [0,1]$ so that the map $\Phi : x \in X \mapsto (f_n(x))_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ is not only continuous but a homeomorphism onto its image. To this end, we will need that for each $n \in \mathbb{N}$, $\Phi(U_n)$ is an open subset of $\Phi(X)$, in other words we will need an open subset V_n of $[0,1]^{\mathbb{N}}$ such that for all $x \in X$, $x \in U_n \Leftrightarrow \Phi(x) \in V_n$.

Let us fix $n \in \mathbb{N}$. If we are given a closed subset F of U_n, Lem. 1.140 provides us $f : X \to [0,1]$ such that $x \notin U_n \Rightarrow f(x) = 1$ and $x \in F \Rightarrow f(x) = 0$. In particular $f(x) < 1 \Rightarrow x \in U_n$. For the converse to hold, we will need to use countably many functions and cover U_n by countably many closed subsets. Let us first check that such a cover exists.

By Hausdorffness for each $x \in U_n$ the singleton $\{x\}$ is closed (Prop. 1.27). Since X is normal and $(U_m)_{m\in\mathbb{N}}$ is a basis for the topology, we then find U_m containing x such that $\overline{U_m} \subseteq U_n$. In other words $\{\overline{U_m} : \overline{U_m} \subseteq U_n\}$ is a cover of U_n.

For each $m \in \mathbb{N}$ such that $\overline{U_m} \subseteq U_n$ we use Lem. 1.140 to pick $f_{n,m} : X \to [0,1]$ continuous such that $f_{n,m}(\overline{U_m}) = \{0\}$ and $f_{n,m}(X \setminus U_n) = \{1\}$. Observe that since the set of $\overline{U_m}$ contained in U_n is a cover of U_n, we now have $x \in U_n$ if and only if there is m such that $f_{n,m}(x) < 1$.

Let I be the set of $(n,m) \in \mathbb{N}^2$ such that $\overline{U_m} \subseteq U_n$. We can now define

$$\begin{aligned}\Phi : X &\to [0,1]^I \\ x &\mapsto (f_{n,m}(x))_{(n,m)\in I}\end{aligned}$$

Since each $f_{n,m}$ is continuous, Φ is continuous. Moreover Φ is injective because if $x \neq y$ we find U_n containing x but not y, and then there is $(n,m) \in I$ such that $x \in \overline{U_m}$ so $f_{n,m}(x) = 0$ but $f_{n,m}(y) = 1$.

Let us finally check that the corestriction of Φ to its image is open: we have $x \in U_n$ if and only if there is m such that $f_{n,m}(x) < 1$, so

$$\Phi(U_n) = \Phi(X) \cap \bigcup_{m\in\mathbb{N}:(n,m)\in I} \pi_{n,m}^{-1}([0,1[)$$

where $\pi_{n,m}$ is the projection onto the (n,m)-coordinate. This implies that $\Phi(U_n)$ is open for the induced topology on $\Phi(X)$, and since (U_n) is a basis for the topology of X, the corestriction of Φ to $\Phi(X)$ is an open map. Moreover Φ is continuous injective, so we conclude that Φ is a homeomorphism onto its image. □

Exercise 1.33. Show directly that every separable metric space (X,d) is homeomorphic to a subspace of $[0,1]^{\mathbb{N}}$ by fixing a dense set $(x_n)_{n\in\mathbb{N}}$ and considering the map $x \mapsto (\min(1, d(x_n,x)))_{n\in\mathbb{N}}$.

1.11 Exercises

1.11.1 Basic exercises

Exercise 1.34. Let (X,d) be a metric space. Fix $x_0 \in X$, and define the SNCF[6] metric by $d_{\mathrm{SNCF}}(x,y) := d(x,x_0) + d(x_0,y)$. Check that this is indeed a metric, and find out why this metric has something to do with the French railway system.

Exercise 1.35. Show that if a topological space admits a countable subbasis then it is second-countable. Deduce that any countable product of second-countable topological spaces is second-countable for the product topology.

Exercise 1.36. Show that the following properties of topological spaces are invariant under homeomorphism: metrizability, separability, first-countability, second-countability, compactness, local compactness, admitting a compatible *complete* metric. Use the latter to show that the topological space $]-1,1[$ admits a compatible complete metric.

Exercise 1.37. Let X be a compact space.

1. Let $\mathcal{U}$ be a family of open subsets of X. Suppose that $\mathcal{U}$ separates points, meaning that for every $x \neq y$ there are disjoint $U, V \in \mathcal{U}$ such that $x \in U$ and $y \in V$. Show that $\mathcal{U}$ generates the topology of X

2. Deduce that every countable compact Hausdorff space is metrizable.

[6] SNCF stands for Société Nationale des Chemins de Fer.

1.11.2 Around Lindelöf's lemmma

Exercise 1.38 (A stronger version of Urysohn's metrization theorem)**.** A topological space is **regular** if points can be separated from closed subsets by open sets: for every closed set F and for every $x \notin F$ there are disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.

1. Show that a topological space X is regular if and only if for every closed subset $F \subseteq X$ and every $x \notin F$ we may find an open set U containing x such that $\overline{U}$ is disjoint from F.
2. Show that given two disjoint closed subsets F and G of a second-countable topological space, we may find a countable cover $(U_n)_{n\in\mathbb{N}}$ of F such that $\overline{U_n}$ is disjoint from G for each $n \in \mathbb{N}$.
3. Show that every regular second-countable topological space is normal. (Hint: apply the previous question to find countable open covers $(U_n)_{n\in\mathbb{N}}$ of F of G with $\overline{U_n} \cap G = \emptyset = \overline{V_n} \cap F$ for all $n \in \mathbb{N}$. Then replace U_n and V_n by smaller open sets so that for all $m \leqslant n$, $V_m \cap U_n = \emptyset$ and $U_m \cap V_n = \emptyset$.)
4. Conclude that every regular second-countable Hausdorff topological space is metrizable. This is often the way Urysohn's metrization theorem is actually stated.

Exercise 1.39. Let X be a second-countable topological space. Show that every basis of the topology of X contains a countable basis.

Exercise 1.40. Let X and Y be topological spaces with X separable. Show that if there is a continuous surjective map $f : X \to Y$ then Y is separable.

Exercise 1.41. Let (X, τ_X) be a Hausdorff topological space, let (Y, τ_Y) be a topological space and $f : X \to Y$. Show that Φ is a homeomorphism onto its image if and only if $\Phi_*\tau_Y = \tau_X$, where $\Phi_*\tau_Y$ is the pullback topology of τ_Y.

Exercise 1.42. Given two reals $a < b$, show directly that the interval $[a, b]$ is compact.

Exercise 1.43. Separability does not pass to closed subspaces.
Consider the space $\{0, 1\}^{[0,1]}$ equipped with the product topology, which we identify to the space of subsets of $[0, 1]$ via their characteristic functions.

1. Show that this space is separable (*Hint:* consider finite unions of open intervals with rational endpoints.)
2. Deduce that this space is not first-countable (*Hint:* Show that any first-countable separable space has at most the cardinality of the continuum).
3. Show that the subspace of singletons is closed and not separable.

1.11.3 Some uses of the axiom of choice in topology

Exercise 1.44 (Alexander's subbase theorem)**.** Let (X, τ) be a topological space, let $\mathcal{U} \subseteq \tau$, suppose that $\mathcal{U}$ generates τ and that every cover of X by elements of $\mathcal{U}$ has a finite subcover. Show that X is compact. (Hint: assume not, show that X has a maximal cover $\mathcal{C}$ without a finite subcover. Show that there is $x \in X$ which is not covered by $\mathcal{U} \cap \mathcal{C}$ and deduce a contradiction)

Exercise 1.45. The Stone-Čech compactification of the integers.

This exercise uses the definition of filters, ultrafilters and their basic properties, for which the reader is refered to Exercise **??**. The **Stone-Čech compactification** of the integers, denoted by $\beta\mathbb{N}$, is the set of all ultrafilters on $\mathbb{N}$ endowed with the topology whose prebase is given by the set of

$$V_A = \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}.$$

where A is a subset of $\mathbb{N}$. In other words, the topology is the topology induced by the product topology on $\{0,1\}^{\mathcal{P}(\mathbb{N})}$.

1. Show that the set of integers (identified to the set of principal ultrafilters) is discrete and dense. Show that the sequence $(n)_{n\in\mathbb{N}}$ has no converging subsequence.

2. Show that the prebase of the topology that we described is actually a base.

3. Let X be a Hausdorff infinite topological space. Show that there is an open set $U \subseteq X$ such that $X \setminus U$ is infinite. Deduce that X contains an infinite discrete subset (an infinite subset onto which the induced topology is the discrete topology).

4. Use this to show that in the Stone-Čech compactification of the integers, every converging sequence is stationary !

Chapter 2

Polish spaces

2.1 Definition and first examples

Definition 2.1. A **Polish space** is a separable topological space whose topology admits a compatible *complete* metric.

The most basic examples of Polish spaces arise as complete separable metric spaces where we forget about the metric and only keep the topology: for instance $\mathbb{R}$ as well as $\mathbb{C}$ form a Polish space for the topology induced by the metric $d(x, y) = |x - y|$. More generally, whenever X is a separable Banach space (a separable normed vector space such that the metric associated to the norm is complete), X is a Polish space for the topology induced by the metric associated to its norm. In particular, we have the following examples of Polish spaces where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

- Every separable Hilbert space over $\mathbb{K}$ is a Polish space for the induced topology.

- Given $p \in [1, +\infty[$ and a countable set X at most countable, the $\mathbb{K}$-vector space $\ell^p(X)$ of maps $X \to \mathbb{K}$ such that $\sum_{x\in X} |f(x)|^p < +\infty$ is a separable Banach space for the norm
$$\|f\|_p := \left(\sum_{x\in X} |f(x)|^p\right)^{1/p},$$
in particular it is a Polish space for the induced topology.

- We will see later that more generally $\mathrm{L}^p(X, \lambda)$ is a separable Banach space whenever X is a Polish space equipped with a σ-finite measure λ (see Chapter **??**).

- Given a compact metrisable space X, the space $\mathcal{C}^0(X, \mathbb{K})$ of continuous functions $X \to \mathbb{K}$ is a separable Banach space for the norm $\|f\|_\infty := \sup_{x\in X} |f(x)|$ (see Exercice **??** for the proof). It is thus a Polish space.

Let us remark that since metrizable topological spaces are separable if and only if they are second-countable (Thm. 1.96), one could take as an equivalent definition that Polish spaces are second-countable topological spaces admitting a compatible complete metric.

We will obtain many more examples of Polish spaces by using the following proposition, often without mentioning it.

Proposition 2.2. *Every closed subspace of a Polish space is Polish for the induced topology.*

Proof. Let X be a Polish space and let $Y \subseteq X$ closed. Then Y is separable by Corollary 1.97, and if we let d be a compatible complete metric on X then its restriction to Y is complete since Y is closed (Prop. 1.78). We conclude that Y is Polish. □

The following lemma is simple but fundamental.

Lemma 2.3. *Every Polish space admits a compatible complete metric which is bounded by* 1.

Proof. Let d be a compatible complete metric on a Polish space X. Then the map $\tilde{d} : (x, y) \mapsto \min(1, d(x, y))$ is a metric on X (this is easily checked directly, but see Exercise **??** for a more general statement). This metric is clearly bounded by 1. Moreover it is compatible with the topology of X since it has the same open balls of radius < 1 as our original metric d. Finally every $\tilde{d}$-Cauchy sequence must be d-Cauchy, so $\tilde{d}$ is complete. □

2.2 Operations on Polish spaces

We have seen that closed subspaces of Polish spaces are Polish. We will now present several other ways of building new Polish spaces. The most important one is countable products.

Proposition 2.4. *Let $(X_i)_{i\in I}$ be a countable family of Polish spaces. Then $\prod_{i\in I} X_i$ is a Polish space for the product topology.*

Proof. By Proposition 1.92 any countable product of separable topological spaces is separable so $\prod_{i\in I} X_i$ is separable.

Since I is countable, we may as well assume $I \subseteq \mathbb{N}$. Using Lemma 2.3 we choose for each $i \in I$ a compatible complete metric d_i on X_i such that $d_i \leqslant 1$. We then define on $\prod_{i\in I} X_i$ a metric d by letting

$$d((x_i), (y_i)) = \sum_{i\in I} \frac{1}{2^i} d_i(x_i, y_i).$$

It is easily checked that d induces the product topology on $\prod_{i\in\mathbb{N}} X_i$ (see exercise **??** for details).

Let $((x_i^n)_{i\in I})_{n\in\mathbb{N}}$ be d-Cauchy. Let $i \in I$. For every $n, m \in \mathbb{N}$ the definition of d yields the inequality

$$d_i(x_i^n, x_i^m) \leqslant 2^i d((x_i^n)_{i\in\mathbb{N}}, (x_i^m)_{i\in\mathbb{N}})$$

so the sequence $(x_i^n)_{n\in\mathbb{N}}$ is d_i-Cauchy . Since d_i is complete, we find a limit $x_i \in X_I$ for the sequence $(x_i^n)_{n\in\mathbb{N}}$. So for every $i \in I$ we have found x_i such that $(x_i^n)_{n\in\mathbb{N}}$ tends to x_i. Since convergence in the product topology is equivalent to convergence in every coordinate (Prop. 1.58), we deduce that $((x_i^n)_{i\in\mathbb{N}})_{n\in\mathbb{N}}$ tends to $(x_i)_{i\in\mathbb{N}}$. This shows that d is complete as wanted. Being separable and having d as a compatible complete metric, the product space $\prod_{i\in I} X_i$ is Polish. □

Remark 2.5. Note that when I is finite we can drop the $\frac{1}{2^i}$ in the definition of d and we don't need to assume each d_i is bounded.

Example 2.6. Since finite products of Polish spaces are Polish and $\mathbb{R}$ is Polish, $\mathbb{R}^n$ is a Polish space. In particular for every $n \in \mathbb{N}$, the space $\mathfrak{M}_n(\mathbb{R})$ ($\simeq \mathbb{R}^{n^2}$) of n-dimensional matrices is Polish. So the group

$$Sl_n(\mathbb{R}) = \{M \in \mathfrak{M}_n(\mathbb{R}) : \det M = 1\}$$

is a Polish space, being closed in $\mathfrak{M}_n(\mathbb{R})$ (indeed it is the preimage of the closed set $\{1\}$ via the determinant function which is continuous because it is a polynomial in the coefficients of the matrix). We will see later that it is actually a Polish group. The same is true of any classical matrix group over $\mathbb{R}$ or $\mathbb{C}$, e.g. $Sl_n(\mathbb{C})$, $Gl_n(\mathbb{C})$ or $SO_n(\mathbb{R})$, but to see that $Gl_n(\mathbb{C})$ is Polish we need to know that open subsets of Polish spaces are Polish (cf. Prop. 2.9)

Example 2.7. We also obtain from the previous proposition two new fundamental infinite-dimensional examples of Polish spaces arising as countable infinite products. These will be the object of the next chapter but they will also play an important role throughout the whole book.

- The **Baire space** is $\mathbb{N}^\mathbb{N}$ equipped with the product topology is a Polish space, viewing $\mathbb{N}$ as a Polish space for the discrete topology.
- The **Cantor space** is $\{0,1\}^\mathbb{N}$ equipped with the product topology is a Polish space, viewing $\{0,1\}$ as a Polish space for the discrete topology. Note that the Cantor space is a closed subspace of the Baire space and that it is compact by Tychonov's theorem.

Two more examples of Polish spaces arising as infinite products are $\mathbb{R}^\mathbb{N}$ and the **Hilbert cube** $[0,1]^\mathbb{N}$. Note that the Hilbert cube is compact by Tychonov's theorem.

Let us now make use of the fact that the product of *two* Polish spaces is Polish so as to obtain that open subspaces of Polish spaces are Polish. The key to this is the following lemma, which we state separately because we will use it again.

Lemma 2.8. *Let X be a Polish space. Every open subset U of X is homeomorphic to a closed subset of $X \times \mathbb{R}$.*

Proof. Let d be a compatible metric on X. Consider the injective map

$$\begin{aligned}\Phi : X &\to X \times \mathbb{R}\\ x &\mapsto (x, \frac{1}{d(x, X \setminus U)})\end{aligned}$$

By Exercise **??** Φ is continuous, and since its inverse is a restriction of the projection $X \times \mathbb{R} \to X$, the map Φ is actually a homeomorphism onto its image.

Moreover its image is closed: let us check this by using the sequential characterisation of closedness. Suppose $(x_n, \frac{1}{d(x_n, X\setminus U)})$ converges to (x, r), then in particular the sequence $\frac{1}{d(x_n, X\setminus U)})$ converges. So $d(x_n, X \setminus U)$ must be bounded away from zero: there is $\delta > 0$ such that $d(x_n, X \setminus U) \geqslant \delta$ for all $n \in \mathbb{N}$. Since $x_n \to x$ we deduce $d(x, X \setminus U) \geqslant \delta$ by continuity. We conclude that $x \in U$ and $r = \frac{1}{d(x, X\setminus U)}$, i.e. $(x, r) \in \Phi(U)$ which is thus closed. $\square$

Proposition 2.9. *Let X be a Polish space and let $Y \subseteq X$ be open. Then Y is Polish for the induced topology.*

Proof. By the previous lemmma Y is homeomorphic to a closed subset of a Polish space. By Prop. 2.2 the latter is Polish so Y is Polish as well. □

We now turn to the main result of this section: countable intersections of open subsets are Polish for the induced topology. Let us first give those subsets a name.

Definition 2.10. A subset A of a topological space X is called G_δ if it can be written as a countable intersection of open subsets: there is a sequence (U_n) of open subsets of X such that $A = \bigcap_{n\in\mathbb{N}} U_n$.

Example 2.11. Let us see some examples of G_δ subsets.

- Every open subset U is G_δ (take $U_n = U$).
- Every countable intersection of G_δ subsets is a G_δ subset (indeed, if $A_n = \bigcap_{m\in\mathbb{N}} U_{n,m}$ then $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{(n,m)\in\mathbb{N}\times\mathbb{N}} U_{n,m}$).
- The set $\mathbb{R}\setminus\mathbb{Q}$ is G_δ in $\mathbb{R}$. More generally in any Hausdorff topological space X, whenever D is a countable subset of X we have that $X\setminus D$ is G_δ. Indeed $X\setminus D = \bigcap_{d\in D} X\setminus\{d\}$ and by Hausdorffness each $X\setminus\{d\}$ is open.
- The set of surjective sequences of integers is G_δ in $\mathbb{N}^\mathbb{N}$ (see Exercise **??** where also more examples of G_δ subsets are provided).

Lemma 2.12. *Every closed subset of a metrizable space is G_δ.*

Proof. Let F be a closed subspace of a metrizable space X equipped with a compatible metric d. For each $\epsilon > 0$ let

$$(F)_\epsilon := \{x \in X : d(x, F) < \epsilon\}.$$

Each $(F)_\epsilon$ is open and $F = \{x \in X : d(x, F) = 0\}$ (see Exercise 1.7). We thus have the equality

$$F = \bigcap_{n\in\mathbb{N}^*} (F)_{1/n},$$

witnessing that F is G_δ. □

We can now generalize to G_δ subsets the fact that closed subsets of Polish spaces are Polish. The proof contains an important idea: we will unfold our countable intersection of open subsets so as to view it as a closed subset of a countable product of open subsets. Let us state this idea as a trick and then prove the announced result.

Trick B (Intersection to product trick)**.** *Let X be a set, let $(A_i)_{i\in I}$ be a family of subsets of X. Consider the injective map Φ which takes $x \in X$ to the constant family equal to x*

$$\begin{aligned}\Phi :&X \to X^I\\ &x \mapsto (x)_{i\in I}.\end{aligned}$$

Then $\Phi\left(\bigcap_{i\in I} A_i\right) = \Phi(X) \cap \prod_{i\in I} A_i.$

Moreover if X is a Hausdorff topological space, $\Phi(X)$ is a closed subset of X^I for the product topology and Φ is a homeomorphism onto its image.

Proof. Observe that if $(x)_{i\in I} \in \prod_{i\in I} A_i$ then by definition for each $i \in I$ we have $x \in A_i$ so $x \in \bigcap_{i\in I} A_i$. The converse clearly holds, so $\Phi^{-1}(\prod_{i\in I} A_i) = \bigcap_{i\in I} A_i$ as stated.

Now by the definition of Φ, the space $\Phi(X)$ is equal to the space Δ_X of constant functions in X^I. The latter is closed in X^I as soon as X is Hausdorff (Exercise 1.15) so $\Phi(X)$ is closed.

Finally let us check that Φ is a homeomorphism onto its image. Clearly Φ is continuous and bijective. Moreover its inverse is a restriction the projection onto the first coordinate, so its inverse is also continuous: we conclude Φ is a homeomorphism onto its image as wanted. □

Theorem 2.13. *Every G_δ subset of a Polish space is Polish for the induced topology.*

Proof. Let X be a Polish space, and let $(U_n)_{n\in\mathbb{N}}$ be a countable family of open subsets of X. Our aim is to show that their intersection $\bigcap_{n\in\mathbb{N}} U_n$ is Polish. We unfold this countable intersection by considering the map

$$\begin{aligned} \Phi : X &\to X^{\mathbb{N}} \\ x &\mapsto (x, x, ...). \end{aligned}$$

from the intersection to product trick B. The map Φ is a homeomorphism onto its image $\Phi(X)$ which is closed and we have

$$\Phi\left(\bigcap_{n\in\mathbb{N}} U_n\right) = \Phi(X) \cap \prod_{n\in\mathbb{N}} U_n.$$

Each U_n is Polish by Proposition 2.9 and so their product $\prod_{n\in\mathbb{N}} U_n$ also is by Proposition 2.4. But $\Phi(X)$ is closed in $X^{\mathbb{N}}$, and the product topology on $\prod_{n\in\mathbb{N}} U_n$ is the topology induced by the product topology in $X^{\mathbb{N}}$ so $\Phi(X) \cap \prod_{n\in\mathbb{N}} U_n$ is closed in $\prod_{n\in\mathbb{N}} U_n$. Using Proposition 2.2 we obtain that $\Phi(X)\cap\prod_{n\in\mathbb{N}} U_n$ is Polish. Since $\bigcap_{n\in\mathbb{N}} U_n$ is homeomorphic to $\Phi(X) \cap \prod_{n\in\mathbb{N}} U_n$ via Φ, we conclude that $\bigcap_{n\in\mathbb{N}} U_n$ is Polish as well. □

Exercise 2.1. Exhibit a compatible complete metric on $\bigcap_{n\in\mathbb{N}} U_n$ by unraveling the proof that $\prod_{n\in\mathbb{N}} U_n$ admits a compatible complete metric.

Let us finally observe that the disjoint union of two Polish topologies is Polish for the disjoint union topology.

Proposition 2.14. *Let X and Y be two Polish spaces. Then $X \sqcup Y$ is Polish for the disjoint union topology.*

Proof. If D_1 is countable dense in X while D_2 is countable dense in Y, it follows from the definition of the disjoint union topology that $D_1 \sqcup D_2$ is dense in $X \sqcup Y$ which is thus separable.

Let d_X, d_Y be compatible complete metrics on X and Y respectively. By Lem. 2.3 we can take these metrics to be bounded by 1. Then let d be the metric which restricts to d_X on X, to d_Y on Y and such that $d(x, y) = 1$ for all $x \in X$ and all $y \in Y$. If (x_n) is a d-Cauchy sequence then let $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m \geqslant N$. By the definition of the metric all the terms starting from $n = N$ belong to the same set.

By symmetry we may as well assume $x_N \in X$, then for each $n \geqslant N$ we must have $x_n \in X$ and $(x_n)_{n\geqslant N}$ is thus a d_X-Cauchy sequence. So (x_n) is convergent for the topology on X, hence it is convergent for the disjoint union topology. We conclude d is complete as wanted. □

Exercise 2.2. Check that the metric we defined above was indeed a compatible metric for the disjoint union topology. Generalize the above result to countable disjoint unions.

2.3 Polish subspaces are exactly G_δ subsets

The following theorem is fundamental and provides a converse to Theorem 2.13.

Theorem 2.15. *Let X be a metrizable topological space, suppose that $Y \subseteq X$ is Polish for the induced topology. Then Y is a G_δ subset of X.*

Proof. Let d_X be a compatible metric on X and d_Y be a compatible complete metric on Y. By Lemma 2.12 the closed set $\overline{Y}$ is G_δ in X. Since G_δ subsets of G_δ subspaces are G_δ in the bigger space, it suffices to show that Y is G_δ is $\overline{Y}$. In other words we may as well assume that $\overline{Y} = X$, i.e. Y is dense in X.

For every $n \in \mathbb{N}$, let U_n be the reunion of the non-empty τ_X-open sets U such that $\operatorname{diam}_{d_Y}(U \cap Y) \leqslant \frac{1}{n}$. We will now prove that $Y = \bigcap_{n\in\mathbb{N}} U_n$. We first point out two immediate facts.

- Every non-empty open $U \subseteq X$ has a non-empty intersection with Y since Y is dense. In particular $\operatorname{diam}_{d_Y}(U \cap Y)$ is alway well-defined.
- Every non-empty open subset $V \subseteq U$ will satisfy $\operatorname{diam}_{d_Y}(V \cap Y) \leqslant \operatorname{diam}_{d_Y}(U \cap Y)$.

Let us show that $Y \subseteq \bigcap_{n\in\mathbb{N}} U_n$: let $y \in Y$. Since the metric d_Y is compatible with the topology induced by X, the d_Y-open ball $B_{d_Y}(y, \frac{1}{2n})$ is equal to $U \cap Y$ for some $U \subseteq X$ open. Since $\operatorname{diam}_{d_Y}(B_{d_Y}(y, \frac{1}{2n})) \leqslant \frac{1}{n}$ we have by definition that U_n contains U so that $y \in U_n$. We conclude that $Y \subseteq \bigcap_{n\in\mathbb{N}} U_n$.

Conversely, let $x \in \bigcap_{n\in\mathbb{N}} U_n$. Then for every $n \in \mathbb{N}$, we find an open neighborhood V_n of x such that $\operatorname{diam}_{d_Y}(V_n \cap Y) \leqslant \frac{1}{n}$. Up to replacing each V_n by a smaller d_X-open ball, we may assume that the sequence (V_n) is decreasing and that $(V_n)_{n\in\mathbb{N}}$ is a neighborhood basis for x in X.

Now pick for every $n \in \mathbb{N}$ some $y_n \in V_n \cap Y$. Observe that (y_n) is d_Y-Cauchy since (V_n) is decreasing and has vanishing d_Y-diameter. So (y_n) has a limit $y \in Y$, but because (V_n) is a decreasing neighborhood basis of x and $y_n \in V_n$ for all $n \in \mathbb{N}$, we also have $y_n \to x$ so $x = y$ and we conclude $x \in Y$.

We thus have the desired reverse inclusion $\bigcap_{n\in\mathbb{N}} U_n \subseteq Y$ and we conclude $Y = \bigcap_{n\in\mathbb{N}} U_n$ as wanted. □

2.4 Every Polish space is homeomorphic to a closed subspace of $\mathbb{R}^\mathbb{N}$

We will now use the techniques and results from previous sections so as to show that every Polish space is homeomorphic to a G_δ subset of $[0,1]^\mathbb{N}$, and then that every Polish space is homeomorphic to a closed subset of $\mathbb{R}^\mathbb{N}$.

Proposition 2.16. *Let X be a Polish space. Then X is homeomorphic to a G_δ subset of $[0,1]^\mathbb{N}$.*

Proof. Since X is separable and metrizable, we know by Exercise 1.33 that X is homeomorphic to a subspace of $[0,1]^\mathbb{N}$. Such a subspace is then Polish and thus a G_δ subset of $[0,1]^\mathbb{N}$ by Theorem 2.15. □

Exercise 2.3. Show that a Polish space is compact if and only if all its compatible metrics are complete.

Let us now use the above proposition to show that every Polish space is homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}}$. Note that such a result could not be true in $[0,1]^{\mathbb{N}}$ for it is compact and thus so are its closed subsets.

Theorem 2.17. *Every Polish space is homeomorphic to a closed subspace of the Polish space* $\mathbb{R}^{\mathbb{N}}$.

Proof. Let X be a Polish space. By Prop. 2.16 we may assume X is a G_δ subset of $[0,1]^{\mathbb{N}}$. Write $X = \bigcap_{n\in\mathbb{N}} U_n$ where each U_n is open in $[0,1]^{\mathbb{N}}$. As in the proof of Thm. 2.13 we use Trick B by considering the map

$$\Phi : X \to \prod_{n\in\mathbb{N}} U_n$$
$$x \mapsto (x, x, ...).$$

The map Φ is a homeomorphism onto its image and its image is a closed subset of $\prod_{n\in\mathbb{N}} U_n$. Now by Prop. 2.8 each U_n is homeomorphic to a closed subset of $[0,1]^{\mathbb{N}} \times \mathbb{R}$ and since $[0,1]^{\mathbb{N}}$ is closed in $\mathbb{R}^{\mathbb{N}}$ we deduce that each U_n is homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}$. We conclude that X is homeomorphic to a closed subset of $(\mathbb{R}^{\mathbb{N}} \times \mathbb{R})^{\mathbb{N}} \cong \mathbb{R}^{\mathbb{N}}$ as desired. □

2.5 Polish compact spaces

We will now improve our characterization of compact metric spaces from Section 1.7.2.3 so as to give various equivalent reformulations of being a compact Polish space.

Theorem 2.18. *Let X be a compact Haudorff topological space. Then the following are equivalent:*

(i) X is Polish,

(ii) X is homeomorphic to a closed subspace of $[0,1]^{\mathbb{N}}$,

(iii) X is metrizable,

(iv) X is second-countable.

Proof. We will prove the equivalence of the above statements by following the chain of implications (i)⇒(iv)⇒(ii)⇒(iii)⇒(i).

We know that Polish spaces are second-countable so (i)⇒(iv).

Let us now show (iv)⇒(ii). Let X be a second-countable Hausdorff compact topological space. Since X is compact Hausdorff, X is normal (Prop. 1.139). So by Urysohn's metrization X is homeomorphic to a subspace of $[0,1]^{\mathbb{N}}$. Since compact subspaces of Hausdorff spaces are closed, we conclude that X is homeomorphic to a closed subspace of $[0,1]^{\mathbb{N}}$ as wanted.

The implication (ii)⇒(iii) follows from the metrizability of $[0,1]^{\mathbb{N}}$ along with the fact that subspaces of metrizable spaces are metrizable.

Finally let us show (iii)⇒(i). Let X be a metrizable compact topological space, then if d is a compatible metric it must be complete and precompact by item (ii) from Thm. 1.116. So we now only need to show that X is separable, which will follow easily from the precompactness of d. Indeed if for every $\epsilon \in \mathbb{Q}^{>0}$ we fix a finite set F_q which is ϵ-dense, then $\bigcup_{q\in\mathbb{Q}^{>0}} F_q$ is a countable dense subset of X. So (i) holds, which ends the proof of the remaining implication (iii)⇒(i). □

Observe that by Tychonov's theorem (Thm. 1.121), the class of compact Polish spaces is closed under countable products. Also any closed subset of a compact Polish space is compact Polish since both properties are inherited by closed subsets (by Prop. 1.105 and Prop. 2.2).

2.6 Polish locally compact spaces

2.7 Baire class 1 functions and semi-continuity

We will now investigate the important notion of Baire class 1 functions, which are functions which fall short of being continuous.

Definition 2.19. Let X and Y be topological spaces, a map $f : X \to Y$ is **Baire class 1** if the preimage of every open subset of Y is an F_σ subset of X.

Remark 2.20. By taking complements we see that equivalently, $f : X \to Y$ is Baire class 1 if the preimage of every closed subset of Y is a G_δ subset of X. Recalling that G_δ subsets of Polish spaces are Polish, this notion will easily provide us examples of Polish spaces.

Example 2.21. By Lem. 2.12, every continuous function between metrizable spaces is Baire class 1.

Remark 2.22. It is straightforward to check that if we compose a continuous function with a Baire class 1 function then we get a Baire class 1 function.

Let us now see the main source of Baire class 1 functions: taking pointwise limits of continuous function (we will see in Prop. **??** that it is sometimes the *only* source of such functions).

Proposition 2.23. *Let X and Y be metrizable topological spaces, with Y separable. Suppose that $f : X \to Y$ is the pointwise limit of a sequence (f_n) of continuous functions, i.e. for all $x \in X$ we have $f(x) = \lim_{n\to+\infty} f_n(x)$. Then f is Baire class 1.*

Proof. We will show that the preimage of every open subset is F_σ. Let U be an open subset of Y. Since Y is metrizable separable we may and do write $U = \bigcup_{n\in\mathbb{N}} B_n$ where each B_n is open and satisfies $\overline{B_n} \subseteq U$.

Observe that if a point $y \in U$ arises as a limit of $y_n \in U$ then since the B_n's are open and cover U we will find some $m, N \in \mathbb{N}$ such that $y_n \in B_m$ for all $n \geqslant N$, in particular $y_n \in \overline{B_m}$ all $n \geqslant N$. Conversely, every limit of elements of $\overline{B_m}$ belongs to U. Applying this to elements of the form $f(x)$, we see that $f(x) \in U$ if and only if there are $m, N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $f_n(x) \in \overline{B}_m$. In other words,

$$f^{-1}(U) = \bigcup_{m\in\mathbb{N}} \bigcup_{M\in\mathbb{N}} \bigcap_{n\geqslant N} f_n^{-1}(\overline{B_m})$$

Since each f_n is continuous, the sets $f_n^{-1}(\overline{B_m})$ are closed, so $f^{-1}(U)$ is F_σ as wanted. □

Exercise 2.4. Identify $2^{\mathbb{N}}$ with the set of subsets $A \subseteq \mathbb{N}$.

1. Show that the maps $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \cap B$ are continuous maps $2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

2. Deduce that the maps $(A_n)_{n\in\mathbb{N}} \mapsto \bigcup_{n\in\mathbb{N}} A_n$ and $(A_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n\in\mathbb{N}} A_n$ are Baire class 1 maps $(2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$. Are they continuous ?

As a concrete example, we will now show that the set of surjective maps $\mathbb{N} \to \mathbb{N}$ is G_δ (hence Polish) by using Baire class 1 functions. This could of course be done directly, but our point is that with Baire class 1 functions we can get easily many more examples.

Lemma 2.24. *Let A be a subset of $\mathbb{N}$. The map $\Phi_A : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ which maps $f \in \mathbb{N}^{\mathbb{N}}$ to $f(A)$ is Baire class 1.*

Proof. Observe that $f(A) = \bigcup_{n\in A} f(\{n\})$. By question 2 of the above exercise, taking a countable union is a Baire class 1 operation so it suffices to show that for each $n \in \mathbb{N}$, the map $\Phi_{\{n\}} : f \mapsto \{f(n)\}$ is continuous. The topology of $2^{\mathbb{N}}$ is generated by open sets of the form $\{A \in \mathbb{N} : k \in A\}$ and $\{A \in \mathbb{N} : k \notin A\}$ for $k \in \mathbb{N}$. By definition $\Phi_n^{-1}(\{A \in \mathbb{N} : k \in A\}) = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) = k\}$ which is clopen while $\Phi_n^{-1}(\{A \in \mathbb{N} : k \notin A\} = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) \neq k\}$ which is open so $\Phi_{\{n\}}$ is indeed continuous. □

Remark 2.25. The same proof shows that for $F \subseteq \mathbb{N}$ finite, the map Φ_F is actually continuous.

Corollary 2.26. *The set of of surjective maps $\mathbb{N} \to \mathbb{N}$ is G_δ in $\mathbb{N}^{\mathbb{N}}$, hence Polish.*

Proof. Using the notation of the above lemma, we have that the set of surjective maps $\mathbb{N} \to \mathbb{N}$ is the preimage of the closed set $\{\mathbb{N}\}$ by the Baire class 1 map $\Phi_{\mathbb{N}}$, so it is G_δ. □

Fundamendal examples of Baire class 1 functions are provided by real-valued semi-continuous functions.

Definition 2.27. Let X be a topological space. A function $f : X \to \mathbb{R}$ is **lower semi-continuous** if for every closed interval of the form $]-\infty, x]$, we have that $f^{-1}(]-\infty, x])$ is closed.

It is **upper semi-continuous** if for every closed interval of the form $[x, +\infty[$ we have that $f^{-1}([x, +\infty[)$ is closed.

Proposition 2.28. *Every lower (resp. upper) semi-continuous function is Baire class 1.*

Proof. If f is lower semi-continuous, then the preimage of every interval of the form $]x, +\infty[$ is open. Observe that an interval of the form $[x, +\infty[$ can be rewritten as a countable intersection

$$[x, +\infty[= \bigcap_{n\in\mathbb{N}} \left]x - \frac{1}{n}, +\infty\right[$$

so by taking the complement the preimage of the set $]-\infty, x[$ is F_σ. Now if $]x, y[$ is an open interval we write it as $]-\infty, y[\cap]x, +\infty[$ and so its preimage is the intersection of two F_σ subsets, hence it is F_σ. Since every open set is a countable reunion of intervals, we conclude that the preimage of every open set is a countable reunion of F_σ sets, so it is F_σ. □

We have the following sequential characterization of semi-continuity, analogous ot that of continuity.

Proposition 2.29. *Let X be a first-countable topological space. Then a function $f : X \to \mathbb{R}$ is lower semi-continuous if and only if for every $x_n \to x$ we have $f(x) \leqslant \liminf_{n\to+\infty} f(x_n)$.*

2.8 The Baire category theorem

Let us now see one of the main features of Polish spaces: countable intersections of dense open subsets are dense. This only uses the fact that Polish spaces admit a compatible complete metric.

Theorem 2.30 (Baire category theorem)**.** *Let X be a topological space which admits a compatible complete metric. Let $(U_n)_{n\in\mathbb{N}}$ be a countable family of dense open subsets of X. Then $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.*

Proof. We need to show that every non-empty open set U intersects $\bigcap_{n\in\mathbb{N}} U_n$. So let U be a non-empty open subset of X. Let us also fix a compatible complete metric d.

Because U_0 is dense, $U_0 \cap U$ is non-empty. Moreover it is an open set, so we may find an open ball $B_0 \subseteq U_0 \cap U$. By shrinking the radius of B_0 if necessary, we may actually assume that $\overline{B_0} \subseteq U_0 \cap U$ and that $\operatorname{diam}_d(B_0) \leqslant 1$

Now $B_0 \cap U_1$ is open non-empty by the density of U_1, so we find an open ball $B_1 \subseteq B_0 \cap U_1$. Again by shrinking its radius we may assume $\overline{B_1} \subseteq B_0 \cap U_1$ and $\operatorname{diam}_d(B_1) \leqslant \frac{1}{2}$.

We continue this construction by induction: assuming that for $n \geqslant 1$ we have build an open ball B_n, then the set $B_n \cap U_{n+1}$ is open non-empty by the density of U_n and we find an open ball $B_{n+1} \subseteq B_n \cap U_{n+1}$. By shrinking its radius we may assume

$$\overline{B_{n+1}} \subseteq B_n \cap U_{n+1} \text{ and } \operatorname{diam}_d(B_{n+1}) \leqslant \frac{1}{2^{n+1}}.$$

Now observe that $(\overline{B_n})_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets of vanishing diameter. Since (X,d) is complete Thm. 1.84 applies. So let $x \in X$ such that $\bigcap_{n\in\mathbb{N}} \overline{B_n} = \{x\}$. Since $\overline{B_0} \subseteq U$, we have $x \in U$, and since for every $n \in \mathbb{N}$ we have $\overline{B_n} \subseteq U_n$, we also have $x \in \bigcap_{n\in\mathbb{N}} U_n$. We conclude that $U \cap \bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$ as wanted. □

Countable intersection of open sets are called G_δ, so the Baire category theorem says that any countable intersection of dense open sets is dense G_δ. The reader should think of dense G_δ sets as big sets in view of the following proposition.

Proposition 2.31. *Let X be a topological space admitting a compatible complete metric. Then every countable intersection of dense G_δ sets is dense G_δ.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be such a sequence of dense G_δ sets, for each $n \in \mathbb{N}$ we find a countable family of open sets $(U_{n,m})_{m\in\mathbb{N}}$ such that $A_n = \bigcap_{m\in\mathbb{N}} U_{n,m}$. So we have $\bigcap_{n,m} U_{n,m} = \bigcap_{n\in\mathbb{N}} A_n$. Each $U_{n,m}$ is dense and since $\mathbb{N}\times\mathbb{N}$ is countable, $\bigcap_{n,m} U_{n,m}$ is dense G_δ by the Baire category theorem. □

Here is a typical application of the Baire category theorem. We will see many more in the exercises, and also once we have more Polish spaces to play with (cf. Chapter 4).

Exercise 2.5. A real number $x \in \mathbb{R}$ is **Liouville** if for every $n \in \mathbb{N}$ there are $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $q > 1$ such that $0 < \left|x - \frac{p}{q}\right| < \frac{1}{q^n}$.

1. Show that every Liouville number is irrational.
2. Show that the set of Liouville numbers is dense G_δ.
3. Deduce that every real number is the sum of two Liouville numbers. (Hint: let L denote the set of Liouville numbers. Show that for every $x \in \mathbb{R}$ the set $x/2 - L \cap -x/2 + L$ is not empty).

Let us note the following reformulation of the Baire category theorem.

Corollary 2.32. *Let X be a topological space which admits a compatible complete metric. Then every countable reunion of closed sets of empty interior has empty interior.*

Proof. This is a direct application of the fact that taking the complement takes closed sets to open sets, countable unions to countable intersections and sets with empty interior to dense sets. □

Example 2.33. The topological space $\mathbb{Q}$ cannot admit a compatible complete metric because it does not satisfy the Baire category theorem, in particular it is not a Polish space. Indeed, consider for each $q \in \mathbb{Q}$ the closed set $\{q\}$. Observe that since every non-empty open subsets of $\mathbb{Q}$ is infinite, $\{q\}$ has empty interior. So if $\mathbb{Q}$ admitted a compatible complete metric, by the Baire category theorem $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ would have empty interior, which is absurd since its interior is $\mathbb{Q}$ (note that here we consider the interior of $\mathbb{Q}$ inside $\mathbb{Q}$ with the induced topology from $\mathbb{R}$, and not inside $\mathbb{R}$!).

2.9 The Cantor-Bendixon rank and perfect Polish spaces

We will now use the Baire category theorem to show that every uncountable Polish space contains a canonical nonempty *perfect* closed subset.

Definition 2.34. Let X be a topological space. A point $x \in X$ is **isolated** in X if $\{x\}$ is open. The space X is **perfect** if it is non-empty and has no isolated point.

Example 2.35. In the set $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ equipped with the induced topology, every point is isolated except for 0.

It is a straightforward exercise to check that every finite Polish space only consists of isolated points and thus is not perfect. Less trivially, the Baire category theorem implies that every countable Polish space must *contain* at least one isolated point. This was used in our proof that $\mathbb{Q}$ is not a Polish space.

Proposition 2.36. *Let X be a countable Polish space. Then X is not perfect.*

Proof. Suppose by contradiction that X has no isolated point. Then for every $x \in X$ the closed set $\{x\}$ has empty interior. But then by the Baire category theorem the countable set $X = \bigcup_{x\in X}\{x\}$ has empty interior, a contradiction. □

Observe that if an open subset of a Polish space contains an isolated point for the induced topology, then the point was already isolated in the whole space. So the above proposition yields that each countable open subset of a Polish space contains an isolated point. By removing all those subsets, we will actually get a canonical closed perfect subspace.

Theorem 2.37. *Let X be an Polish space. Then there is a closed perfect subspace of X which contains every Polish perfect subspace of X. Its complement is the reunion of the open countable subsets of X, which is countable.*

Proof. Consider the open set

$$U := \bigcup\{V \subseteq X \text{ open countable}\}.$$

Let us show that U is countable. Since X is Polish it has a countable basis $\mathcal{B}$ of open sets. So every $V \subseteq X$ open countable is a reunion of elements of $\mathcal{B}$ which must be themselves countable, and we conclude that

$$U := \bigcup\{V \in \mathcal{B} : V \text{ is countable}\}.$$

So U is countable, being a union of countably many countable sets.

Let $x \in X \setminus U$, then by the definition of U every neighborhood V of x is uncountable, and since U is countable this implies $V \cap X \setminus U$ is uncountable. In particular x is not isolated in $X \setminus U$ for the induced topology. We conclude that $X \setminus U$ is perfect. Let us show that $X \setminus U$ is the biggest Polish perfect subspace of X.

Suppose Y is another perfect Polish subspace of X. Then $U \cap Y$ is a countable Polish space. Suppose by contradiction it is nonempty. Then it contains an isolated point for the induced topology by the above proposition . Since U is open we conclude that $U \cap Y$ contains an isolated point for Y which is thus not perfect, a contradiction. So $Y \cap U$ is empty, in other words $Y \subseteq X \setminus U$ as wanted. □

The maximum perfect Polish subspace of a Polish space X is called its **perfect kernel**.

Remark 2.38. By the above theorem, the complement of the perfect kernel is countable. So when X is uncountable its perfect kernel is nonempty. Moreover when X is countable its perfect kernel is trivial by Proposition 2.36.

Note that since the complement of the perfect kernel is the reunion of the countable open subsets, the the perfect kernel of X is the set of all points $x \in X$ such that every neighborhood of x is uncountable. These are called **condensation points**.

We now present another perhaps more intuitive way of obtaining a perfect closed subset of a Polish space : we will keep removing isolated points until we cannot anymore.

Definition 2.39. Let X be a topological space. Its **Cantor-Bendixon derivative** is the subset X' defined by

$$X' = X \setminus \{x \in X : x \text{ is isolated in } X\}.$$

Note that by definition X is perfect if and only if $X' = X$. Moreover X' is closed in X because its complement is a reunion of open singletons. We can now define by induction on countable ordinals the α'th Cantor-Bendixon derivative $X^{(\alpha)}$ of a topological space X as follows:

- $X^{(0)} = X$;
- $X^{(\alpha+1)} = (X^{(\alpha)})'$;
- $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ when α is a limit ordinal.

Observe that since intersections of closed sets are closed, each $X^{(\alpha)}$ is closed.

Proposition 2.40. *Let X be a Polish space. There is a countable ordinal β such that $X^{(\beta)}$ is a perfect closed subset of X. Moreover $X^{(\beta)}$ is then the perfect kernel of X.*

Proof. Define $X^{(\infty)} = \bigcap_{\alpha\in\omega_1} X^{(\alpha)}$. Then since each $X^{(\alpha)}$ is closed, $(X \setminus X^{(\alpha)})_{\alpha\in\omega_1}$ is an open cover of $X \setminus X^{(\infty)}$. By Lindelöf's lemma, it contains a countable subcover $(X \setminus X^{(\alpha_n)})_{n\in\mathbb{N}}$, which by taking complements again means $X^{(\infty)} = \bigcap_{n\in\mathbb{N}} X^{(\alpha_n)}$.

Since ω_1 is stable under countable supremums we can define $\beta = \sup\{\alpha_n : n \in \mathbb{N}\} \in \omega_1$. We then have $X^{(\beta)} = X^{(\infty)}$. In particular $X^{(\beta)} = X^{(\beta+1)}$, so $X^{(\beta)}$ is a perfect closed subset of X as wanted.

Moreover at each step we only remove countably many points, so since β is countable the complement of $X^{(\beta)}$ is a countable open subset of X. Theorem 2.37 thus yields that $X^{(\beta)}$ contains the perfect kernel of X, and since $X^{(\beta)}$ is perfect we conclude that they are equal. □

Definition 2.41. The **Cantor-Bendixon rank** of a Polish space X is the least ordinal $\alpha \in \omega_1$ such that $X^{(\alpha)}$ is perfect. It is denoted by $CB(X)$.

Exercise 2.6. Show that for every $\alpha \in \omega_1$, there is a Polish space X such that $CB(X) = \alpha$. (Hint: for the limit case use the disjoint union operation and in the successor case multiply by $\mathbb{N} \cup \{\infty\}$).

2.10 Exercises

Exercise 2.7 (difficult)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that for all $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $f^{(n)}(x) = 0$. Then f is a polynomial.

Exercise 2.8. Show that a Polish space is compact if and only if *all* its compatible metrics are complete.

A countable family of pseudometric which separates points yields a metric.
Redo open is Polish via $x \mapsto (x, \frac{1}{d(x,X\setminus Y)})$.
Redo countable product by hand.

Chapter 3

The Cantor space, the Baire space and schemes

We will now introduce the two most fundamental examples of Polish spaces from the theoretical point of view, namely the Cantor space and the Baire space. The first will be surjectively universal among all compact Polish spaces, while the second will be surjectively universal among all Polish spaces, in a sense that we will define later.

3.1 The Cantor space

Definition 3.1. The **Cantor space** is the space $2^{\mathbb{N}} := \{0,1\}^{\mathbb{N}}$ of infinite sequences of zeros and ones equipped with the product topology, viewing $\{0,1\}$ as a topological space for the discrete topology.

Identifying a subset of $\mathbb{N}$ to its characteristic function, we may also view the Cantor space as the space of subsets of $\mathbb{N}$. Note that by the definition of the product topology, a sequence (A_n) of subsets of $\mathbb{N}$ converges to $A \subseteq \mathbb{N}$ if and only if for every $k \in \mathbb{N}$ and for all large enough $n \in \mathbb{N}$ we have $k \in A_n$ if and only if $k \in A$.

In the next proposition, recall that a topological space is called *zero-dimensional* if its topology admits a basis made of *clopen* sets (i.e. sets which are both closed and open).

Proposition 3.2. *The Cantor space is a compact zero-dimensional Polish space.*

Proof. The space $\{0,1\}$ is compact zero-dimensional and Polish. Since the class of zero-dimensional compact Polish spaces is stable under countable products (see Prop. **??** and Prop. **??**), the Cantor space $\{0,1\}^{\mathbb{N}}$ itself is a compact zero-dimensional Polish space. □

We will now construct an explicit basis of clopen sets for the topology of the Cantor space. In order to do so, we introduce some important terminology.

We denote by $2^{<\mathbb{N}}$ the set of finite sequences of zeros and ones, i.e. tuples of the form $s = (s_0, ..., s_{n-1})$ where $x_i \in \{0,1\}$ for all $i \in \{0, ..., n-1\}$ and $n \in \mathbb{N}$. To lighten the notation, will also write these tuples as *words*, so we define $s_0...s_{n-1} = (s_0, ..., s_{n-1})$.

The unique integer $n \in \mathbb{N}$ such that $s \in 2^{<\mathbb{N}}$ may be written as $s = s_0...s_{n-1}$ is called the **length** of s and denoted by $|s|$. The unique sequence of length zero is denoted by $\emptyset$.

For a finite sequence $s \in 2^{<\mathbb{N}}$ and $m \leqslant |s|$, we let $s_{\restriction m} = s_0...s_{m-1}$. For an infinite sequence $x \in 2^{\mathbb{N}}$, we let $x_{\restriction n} = (x_0, ..., x_{n-1})$. Now every finite sequence $s \in 2^{<\mathbb{N}}$ defines a set $N_s \subseteq 2^{\mathbb{N}}$ given by

$$N_s := \{x \in 2^{\mathbb{N}} : x_{\restriction |s|} = s\}.$$

The following proposition justifies the denomination of sets of the form N_s as **basic clopen sets**.

Proposition 3.3. *The family $(N_s)_{s\in 2^{<\mathbb{N}}}$ is a countable basis for the topology of the Cantor space which consists of clopen sets.*

Proof. Let $s \in 2^{<\mathbb{N}}$. Note that N_s is the preimage of the clopen set $\{s\} \subseteq \{0,1\}^{|s|}$ under the projection onto the first $|s|$ coordinates, so N_s is clopen.

Moreover for every $x \in 2^{\mathbb{N}}$ the definition of the product topology implies that the family of clopen sets $(N_{x\restriction_n})_{n\in\mathbb{N}}$ is a neighborhood basis of x. As a consequence the whole family $(N_s)_{s\in 2^{<\mathbb{N}}}$ is a basis for the topology of the Cantor space. □

Note that for $s, t \in 2^{<\mathbb{N}}$, we have $N_s \subseteq N_t$ if and only if t is an **initial segment** of s, meaning $|t| \leqslant |s|$ and $s_{\restriction |t|} = t$, which we also write as $s \leqslant t$.

⚠Note that for this order, $s < t$ implies that s is a *longer* word than t !

We can represent the order $<$ as a *tree-order* as given by the following picture, where we observe that $s < t$ if and only if s is a descendant of t. The empty sequence is the *root* of the tree.

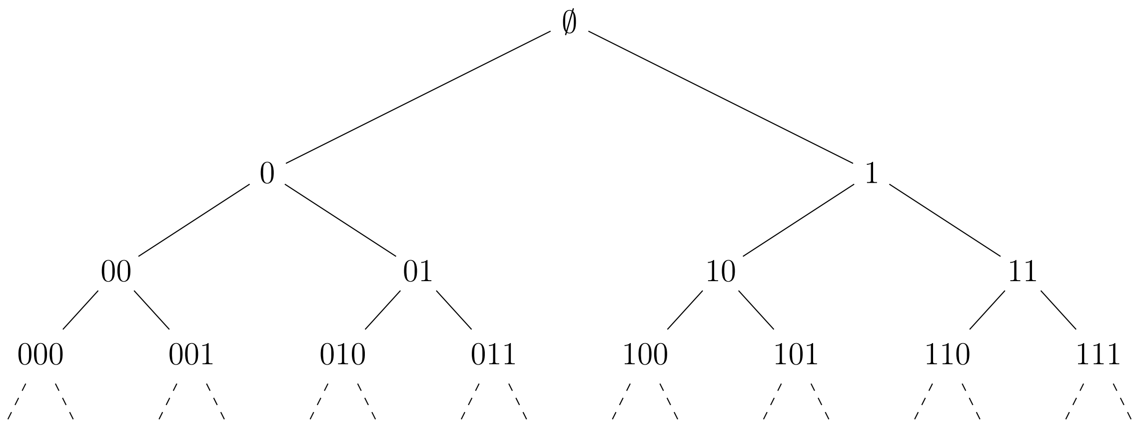

Figure 3.1: The rooted tree structure on $2^{<\mathbb{N}}$

For this to work, we first need to show that every uncountable Polish space contains a non-empty perfect closed subset, which is the point of the next section.

3.2 Polish spaces satisfy the continuum hypothesis topologically

In this section we prove that every uncountable Polish space contains a copy of the Cantor set. A key step will be to find in any uncountable Polish space a family $(U_s)_{s\in 2^{<\mathbb{N}}}$ of open subsets which looks very much alike the N_s. Here is a precise definition.

Definition 3.4. A **Cantor scheme** on a topological space X is a family $(U_s)_{s\in 2^{<\mathbb{N}}}$ of subsets of X such that for every $s \in 2^{<\mathbb{N}}$ and every $i \in \{0,1\}$ we have $\overline{U_{si}} \subseteq U_s$.

Observe that whenever $f : 2^{\mathbb{N}} \to X$ is a continuous map, the family $(f(N_s))_{s\in 2^{<\mathbb{N}}}$ is a Cantor scheme (to see this note that $f(N_s)$ is closed by compactness and continuity). We will now find conditions which recast the injectivity and continuity of f in terms of the Cantor scheme.

Definition 3.5. A Cantor scheme $(U_s)_{s\in 2^{<\mathbb{N}}}$ on a topological space X is **proper** if for every $s \in 2^{<\mathbb{N}}$ we have $U_{s0} \cap U_{s1} = \emptyset$.

Note that if a Cantor scheme is proper then $s \not\leqslant t$ and $t \not\leqslant s$ implies $U_s \cap U_t = \emptyset$. In particular if $|t| = |s|$ but $s \neq t$, then $U_s \cap U_t = \emptyset$.

Definition 3.6. If (X, d) is a metric space, we say that the Cantor scheme $(U_s)_{s\in 2^{<\mathbb{N}}}$ on X is has **vanishing diameter** if for every $x \in 2^{\mathbb{N}}$ we have $\operatorname{diam}_d(U_{x\restriction n}) \to 0$.

Exercise 3.1. Show that every continuous map $f : 2^{\mathbb{N}} \to X$ yields a Cantor scheme $(f(N_s))_{s\in 2^{<\mathbb{N}}}$ of vanishing diameter which is proper if and only if f is injective.

Proposition 3.7. *Let (X, d) be a complete metric space, let $(U_s)_{s\in 2^{<\mathbb{N}}}$ be a convergent Cantor scheme consisting of nonempty sets. Then there is a continuous map $f : 2^{\mathbb{N}} \to X$ such that for every $x \in 2^{\mathbb{N}}$,*

$$\{f(x)\} = \bigcap_{n\in\mathbb{N}} \overline{U_{x\restriction n}} = \bigcap_{n\in\mathbb{N}} U_{x\restriction n}.$$

Moreover if the scheme is proper then f is injective.

Proof. Let $x \in 2^{\mathbb{N}}$. Observe that $(\overline{U_{x\restriction n}})$ is a decreasing family of nonempty closed subsets of vanishing diameter, so by Theorem 1.84 its intersection is a singleton. So we can indeed define a map $f : 2^{\mathbb{N}} \to X$ by the equation $\{f(x)\} = \bigcap_{n\in\mathbb{N}} \overline{U_{x\restriction n}}$. Moreover by condition (a) of d-convergence, we have for each n the inclusions $\overline{U_{x\restriction n+1}} \subseteq U_{x\restriction n} \subseteq \overline{U_{x\restriction n}}$ so we also have $\{f(x)\} = \bigcap_{n\in\mathbb{N}} U_{x\restriction n}$.

Let us now see why f is continuous: fix $\epsilon > 0$ and $x \in 2^{\mathbb{N}}$, then by condition (b) of d-convergence there is $N \in \mathbb{N}$ such that $\operatorname{diam}_d(U_{x\restriction N} < \epsilon$. But for all $y \in N_{x\restriction N}$ we have by definition $f(y) \in U_{y\restriction N} = U_{x\restriction N}$ so $d(f(x), f(y)) \leqslant \epsilon$. So f is continuous.

If the scheme is proper and x and y are distinct elements of the Cantor space, there is $n \in \mathbb{N}$ such that $x\restriction_n \neq y\restriction_n$. Then $U_{x\restriction n}$ and $U_{y\restriction n}$ are disjoint, and the first contains $f(x)$ while the second contains $f(y)$ which are thus also distinct. So properness implies injectivity as wanted. □

Theorem 3.8. *Let X be a perfect Polish space. Then there is a closed subset of X which is homeomorphic to the Cantor space.*

Proof. Let X be a perfect Polish space and let d be a compatible complete metric on X. Since $2^{\mathbb{N}}$ is compact Hausdorff, every continuous injective map $2^{\mathbb{N}} \to X$ must be a homeomorphism onto its image, so we only need to find a continuous injective map $f : 2^{\mathbb{N}} \to X$.

To this end, we will define by induction a scheme $(U_s)_{s\in 2^{<\mathbb{N}}}$ consisting of non-empty open subsets of X such that:

(i) For every $s \in 2^{<\mathbb{N}}$ we have $\overline{U_{s0}} \sqcup \overline{U_{s1}} \subseteq U_s$

(ii) For every $s \in 2^{<\mathbb{N}}$ we have $\operatorname{diam}_d(U_s) < 2^{-|s|}$

Observe that condition (i) implies that the scheme is a proper Cantor scheme while condition (ii) implies that it has vanishing diameter. By the above proposition this scheme yields the desired continuous injective map. So we only need to explain how to build such a scheme.

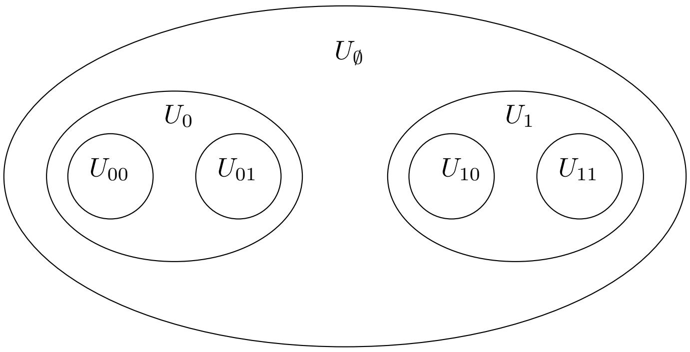

Figure 3.2: The beginning of our Cantor scheme on X

We start with $U_\emptyset$ an open ball of diameter less than 1 and then, assuming U_s has been defined, we define U_{s0} and U_{s1} as follows.

Since U_s is open and X is perfect, we have that U_s is perfect and since it is not empty, it contains at least two distinct points x and y. We then find disjoint open balls U_{s0} and U_{s1} around x and y respectively, and up to shrinking their radiuses we have $\overline{U_{s0}} \sqcup \overline{U_{s1}} \subseteq U_s$, $\operatorname{diam}_d(U_{s0}) < 2^{-|s|-1}$ and $\operatorname{diam}_d(U_{s1}) < 2^{-|s|-1}$ as wanted. □

Remark 3.9. Observe that if we endow the Cantor space with the compatible metric defined by $d(x, y) = 2^{-n(x,y)}$ where $n(x, y) = \min\{n \in \mathbb{N} : x_n \neq y_n\}$, then by condition (ii) the map associated to the Cantor scheme is actually 1-Lipschitz.

We can finally apply the previous theorem to get the following remarkable result: Polish spaces satisfy the continuum hypothesis in a stronger topological way.

Corollary 3.10. *Let X be an uncountable Polish space. Then X contains a closed subset which is homeomorphic to the Cantor space.*

Proof. By Theorem 2.37, the uncountability of X yields that its perfect kernel Y is closed nonempty. We now apply the previous theorem to Y and get the desired result. □

3.3 The Cantor space surjects onto every compact metrizable space

We now use again Cantor schemes to show that the Cantor space surjects continuously onto every compact Polish space.

Theorem 3.11. *Let X be a non empty compact Polish space. Then there is a continuous surjection $f : 2^{\mathbb{N}} \to X$.*

Proof. Let X be a compact Polish space and let d be a compatible complete metric on X. Let us build by induction a Cantor scheme $(F_s)_{s\in 2^{<\mathbb{N}}}$ consisting of nonempty closed sets such that

(i) $F_\emptyset = X$

(ii) For all $s \in 2^{<\mathbb{N}}$, $F_s = F_{s0} \cup F_{s1}$

(iii) The scheme has vanishing diameter

We must start with $F_\emptyset = X$. By compactness, we can cover X by a finite family $(F_i)_{i=1}^n$ of closed balls of diameter less than $1/2$. We now put $F_{0^i} = X_i \cup \dots \cup X_n$ for $i \leqslant n$ and $F_{0^i 1} = X_i$ for $i < n$ (see Figure **??**). The construction is carried on by repeating this process within each X_i using closed sets of diameter $< 1/3$, and so on.

As in the proof of Thm. 3.8, we then get from our Cantor scheme a continuous map $f : 2^{\mathbb{N}} \to X$ defined by $\{f(x)\} = \bigcap_{n\in\mathbb{N}} \overline{U_{x\restriction n}}$. Let us check that f is surjective. If $y \in X$, we define by induction a sequence $(x_n)_{n\in\mathbb{N}} \in 2^{\mathbb{N}}$ by letting x_n be the first element of $\{0,1\}$ such that $y \in U_{x_0 \dots x_n}$. Observe that conditions (i) and (ii) ensure that (x_n) is well defined. We then have $y = f((x_n))$ by definition so f is surjective. □

3.4 A characterization of the Cantor space as a zero-dimensional space

3.5 Trees and their boundaries

Before we move on to the Baire space, it is useful to formalize the descriptive set-theoretic notion of *tree*. For this we first extend the notation we used for indexing Cantor schemes, replacing $\{0,1\}$ by an arbitrary set A.

Let $A^{<\mathbb{N}} = \bigsqcup_{n\in\mathbb{N}} \mathbb{N}^n$ the set of finite sequences of elements of A. Given $s \in A^{<\mathbb{N}}$, the unique integer $n \in \mathbb{N}$ such that $s \in A^n$ is the **length** $|s|$ of s. Given $s \in A^{<\mathbb{N}}$ and $n \leqslant |s|$, we let $s_{\restriction n} = (s_0, ..., s_{n-1})$. Say that t is an **initial segment** of s if $|t| \leqslant |s|$ and $s_{\restriction |t|} = t$. We then write $s \leqslant t$ and also say that s is a **descendant** of t or that t is a **parent** of s. The order $\leqslant$ is called the **tree order** on $A^{<\mathbb{N}}$.

⚠Note that for this order, $s < t$ implies that s is a *longer* word than t !

Definition 3.12. A **tree** (on a set A) is a subset $T \subseteq A^{<\mathbb{N}}$ such that whenever $s \in T$, all the parents of s also belong to t: for all $t \in A^{<\mathbb{N}}$, if $s < t$ then $t \in T$.

The elements of a tree are called its **nodes**.

Definition 3.13. Let T be a tree, then a node $s \in T$ is a **leaf** if there is no $t \in T$ such that $t < s$.

So by definition leafs are minimal elements for the tree-order on T, or equivalently they are nodes without descendants.

A tree on a set A is **pruned** if it has no leaf. Note that this is equivalent to saying that for every $s \in T$, there is $a \in A$ such that $sa \in T$ (indeed if we have some $t \in T$ such that $t < s$ then $t_{\restriction |s|+1} \in T$).

We denote by $\mathrm{Pr}(A)$ the set of pruned trees on A.

For an infinite sequence $x \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$, we let $x_{\restriction n} = (x_0, ..., x_{n-1})$.

Definition 3.14. An **infinite branch** or an **end** of a tree T is a sequence $x \in A^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have $x_{\restriction n} \in T$.

Let us see that in a pruned tree, every node is the beginning of an end (!).

Proposition 3.15. *Let T be a pruned tree, let $s \in T$. Then there is an end x of T such that $x_{\restriction |s|} = s$.*

Proof. Let T be a nonempty pruned tree. We define the infinite branch $(x_n)_{n\in\mathbb{N}}$ by induction, starting of course with $x_0 \cdots x_{|s|-1} = s$. For $n \geqslant |s|$, having constructed $x_0, ..., x_{n-1} \in A$ so that $x_0 \cdots x_{n-1} \in T$, we simply use that T is pruned so as to find $x_n \in A$ such that $x_0 \cdots x_{n-1}x_n \in T$. □

Corollary 3.16. *Every nonempty pruned tree has an end.* □

Remark 3.17. The reader may have noticed that we used the axiom of dependent choice, which is actually equivalent to the statement that for every set A, every pruned tree on A has an infinite branch (see Exercise **??**).

We now give another well-known sufficient condition for having an infinite branch.

Definition 3.18. A tree T on a set A is **locally finite** if every $s \in T$ has only finitely many direct descendants: there are only finitely many $a \in A$ such that $sa \in T$.

Proposition 3.19 (König's lemma)**.** *Every infinite locally finite tree has an end.*

Proof. Let T be an infinite locally finite tree, consider the subtree

$$S := \{s \in T : s \text{ has infinitely many descendents in } T\}.$$

Let $s \in S$, let $sa_1, ..., sa_n$ be the direct descendants of s. Then one of $sa_1, ..., sa_n$ must have infinitely many descendants because otherwise s would have finitely many descendants, contradicting the definition of S. So S is pruned.

Moreover since T is infinite the empty sequence belongs to S which is thus nonempty. By the above proposition S has an infinite branch. Such an infinite branch is also an infinite branch for T. □

Let us now study the connection between trees and the product topology of $A^{\mathbb{N}}$ where A is equipped with the discrete topology. For $s \in A^{<\mathbb{N}}$ we let

$$N_s := \{x \in A^{\mathbb{N}} : x_{\restriction |s|} = s\}.$$

Observe that each N_s is clopen and that $(N_s)_{s\in A^{<\mathbb{N}}}$ forms a basis for the topology of $A^{\mathbb{N}}$.

Definition 3.20. Let T be a tree. Its **boundary** is defined as the set ∂T of ends of T.

Lemma 3.21. *Let T be a tree. Then its boundary ∂T is a closed subset of $A^{\mathbb{N}}$.*

Proof. By definition $x \notin \partial T$ means that there is $n \in \mathbb{N}$ such that $x_{\restriction n} \notin T$. Then $N_{x_{\restriction n}}$ is an open neighborhood of x which is disjoint from ∂T. We conclude ∂T is closed. □

We have the following important further connection between closed subsets of $A^{\mathbb{N}}$ and *pruned* trees.

Proposition 3.22. *The map $\partial : T \mapsto \partial T$ induces a bijection between the set of pruned trees and the set of closed subsets of $A^{\mathbb{N}}$. Its inverse is the map*

$$F \mapsto T_F := \{s \in A^{<\mathbb{N}} : N_s \cap F \neq \emptyset\}.$$

Proof. We have already seen that ∂ takes values into closed subsets of $A^{\mathbb{N}}$. Moreover if F is a closed subset, T_F is pruned: if $s \in A^{<\mathbb{N}}$ and $N_s \cap F \neq \emptyset$ we have $F \cap N_s = \bigcup_{a \in A} F \cap N_{sa}$, so there is $a \in A$ such that $N_{sa} \cap F \neq \emptyset$.

To finish the proof, it suffices to show that for every pruned tree T we have $T_{\partial T} = T$ and that for every closed subset F we have $\partial T_F = F$.

So let T be a pruned tree, observe that $T_{\partial T}$ is the set of $s \in A^{<\mathbb{N}}$ such that there is an end x of T satisfying $x_{\restriction |s|} = s$. In other words $T_{\partial T}$ is the set of beginnings of ends of T, which is equal to T by Prop. 3.15.

Now let F be a closed subset of $A^{\mathbb{N}}$, by definition ∂T_F is the set of $x \in A^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have $N_{x_{\restriction n}} \cap F \neq \emptyset$. Since for every $x \in A^{\mathbb{N}}$ the family $(N_{x_{\restriction n}})_{n \in \mathbb{N}}$ is a neighborhood basis for x, we conclude that ∂T_F is the closure of F, so $\partial T_F = F$ as wanted. □

3.6 An application to the Baire space

By definition, the Baire space is the space $\mathbb{N}^{\mathbb{N}}$ equipped with the product topology. Using the notation from the previous section, we have that $(N_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ is a basis for its topology consisting of clopen sets, in particular the Baire space is a zero-dimensional Polish space.

We will now use the correspondence between pruned trees and closed subsets so as to build a nice choice function for closed sets.

Proposition 3.23. *Let T be a nonempty pruned tree on $\mathbb{N}$. Then T has an end x such that for every $s \in T$ we have $x_i \leqslant s_i$ for every $i \in \{0, ..., |s| - 1\}$.*

Proof. We construct the end by induction, starting with $x_0 = \min\{k \in \mathbb{N} : k \in T\}$ and then, $x_0, ..., x_n$ having been built, we let $x_{n+1} = \min\{k \in \mathbb{N} : x_0 \cdots x_n k \in T\}$. □

Definition 3.24. The end of T constructed in the above proposition is called the **leftmost end** (or **leftmost branch**) of T. It is denoted by $l(T)$.

Identifying a nonempty closed subset F of $\mathbb{N}^{\mathbb{N}}$ to the corresponding nonempty pruned tree $T_F = \{s \in \mathbb{N}^{<\mathbb{N}} : N_s \cap F \neq \emptyset\}$ via Proposition 3.22, we also let $l(F) := l(T_F)$ be the **leftmost end** of the closed set F. Observe that $l(F) \in F$.

We will now use the leftmost end to show that every nonempty closed subset F of $\mathbb{N}^{\mathbb{N}}$ is a continuous image of $\mathbb{N}^{\mathbb{N}}$ via a map which is moreover the identity on F.

Theorem 3.25. *Let F be a nonempty closed subset of $\mathbb{N}^{\mathbb{N}}$. Then there is a continuous map $f : \mathbb{N}^{\mathbb{N}} \to F$ such that $f(x) = x$ for every $x \in F$.*

Proof. For $x \in \mathbb{N}^{\mathbb{N}} \setminus F$, let $k(x) = \max\{k \in \mathbb{N} : N_{x \restriction k} \cap F \neq \emptyset\}$. Observe that k is a continuous function on $\mathbb{N}^{\mathbb{N}} \setminus F$ because it is locally constant (indeed if we fix $x_0 \in \mathbb{N}^{\mathbb{N}} \setminus F$ we see that $k(x) = k(x_0)$ for every $x \in N_{x_0 \restriction k(x_0)+1}$).

We then define $f : \mathbb{N}^{\mathbb{N}} \to F$ by

$$f(x) = \begin{cases} x & \text{if } x \in F \\ l(N_{x_{\restriction k(x)}} \cap F) & \text{if } x \notin F. \end{cases}$$

Since k is locally constant, the function f is locally constant and hence continuous when restricted to the open set $\mathbb{N}^{\mathbb{N}} \setminus F$. So to conclude that f is continuous, we only need to show that for every $x \in F$ we have $f(y) \to f(x)$ when $y \to x$.

So let V be a neighborhood of some $x \in F$, up to shrinking V we may assume that $V = N_{x\restriction N}$ for some $N \in \mathbb{N}$. Then for all $y \in N_{x\restriction N}$, either $y \in F$ and then $f(y) = y \in N_{x\restriction N}$ or $y \notin F$, in which case $k(y) \geqslant N$ since $F \cap N_{y\restriction N} = F \cap N_{x\restriction N} \neq \emptyset$ and thus $f(y) = l(N_{y\restriction k(y)} \cap F) \in N_{x\restriction N}$. In both cases we have $f(y) \in V$ as wanted. □

Remark 3.26. We will see in Corollary **??** that the same is true if we replace the Baire space $\mathbb{N}^\mathbb{N}$ by any zero-dimensional Polish space. For now, we can remark that the above proof also works for the Cantor space.

Exercise 3.2. Show that the map f from the above proof is actually 1-Lipschitz for the metric defined by $d(x, y) = 2^{-n(x,y)}$ where $n(x, y) = \min\{n \in \mathbb{N} : x_n \neq y_n\}$.

3.7 Around the universality of the Baire space

In this section, we will see various results which show the ubiquity of the Baire space. A key tool for these results will be the following natural analogue of Cantor schemes.

Definition 3.27. A **Luzin scheme** on a set X is a family $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of X such that for every $s \in \mathbb{N}^{<\mathbb{N}}$ and every $i \in \mathbb{N}$ we have $\overline{F_{si}} \subseteq F_s$.

Observe that a family $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of X is a Luzin scheme if and only if for every for every $s, t \in \mathbb{N}^{<\mathbb{N}}$ such that $t < s$ we have $\overline{F_t} \subseteq F_s$. Just like with Cantor schemes, we can define a condition which will allows us to produce a continuous map from a Luzin scheme (convergence), and a further condition which makes the map injective (properness).

Definition 3.28. If (X, d) is a metric space, we say that the Luzin scheme $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ on X is d-**convergent** if the following two conditions are satisfied:

(a) for every $s \in \mathbb{N}^{<\mathbb{N}}$ and every $i \in \{0, 1\}$ we have $\overline{F_{si}} \subseteq F_s$

(b) for every $x \in \mathbb{N}^\mathbb{N}$ we have $\operatorname{diam}_d(F_{x\restriction n}) \to 0$.

Definition 3.29. A Luzin scheme on a set X is **proper** if

We emphasize again that if a Luzin scheme is proper then $s \not\leqslant t$ and $t \not\leqslant s$ implies $U_s \cap U_t = \emptyset$, in particular if $|t| = |s|$ but $s \neq t$, then $U_s \cap U_t = \emptyset$.

Theorem 3.30. *Let X be a Polish space. Then there is a closed subset F of $\mathbb{N}^\mathbb{N}$ and a continuous bijection $f : F \to X$ whose inverse f^{-1} is Baire-class 1.*

Proof. We will build a proper Lusin scheme on X

□

Theorem 3.31. *Let X be a Polish space. Then there is an open continuous surjection $\pi : \mathbb{N}^\mathbb{N} \to X$.*

Proof. □

The above result has many applications, but for now we shall content ourselves with the following important way of recognising a Polish space which is due to Sierpinski.

Theorem 3.32. *Let X be a Polish space and let Y be a metrizable space. Suppose there is a continuous open map $\pi : X \to Y$. Then Y is Polish.*

Proof. Being the continuous image of a separable topological space, Y is separable (see Ex. **??**).

Observe that by Theorem 3.31, we may as well assume $X = \mathbb{N}^{\mathbb{N}}$. We thus get a decreasing Souslin scheme (Y_s) on Y consisting of open sets given by $Y_s = \pi(N_s)$

Let d be a compatible metric on Y. Since π is continuous, this Souslin scheme has vanishing diameter. Let $\hat{Y}$ be the completion of (Y, d) where we still write the associated metric as d. We get another decreasing Souslin scheme (U_s) on $\hat{Y}$ by letting U_s be the interior of the closure of Y_s in $\overline{Y}$. Note that (U_s) still has vanishing diameter and $V_s \subseteq U_s$ so that for all $x \in \mathbb{N}^{\mathbb{N}}$ we have $\bigcap_{n\in\mathbb{N}} U_{x\restriction n} = \{\pi(x)\}$. If (U_s) were locally finite, we would be done by a direct application of Lem. **??**

We will thus "refine" (U_s) so as to obtain a locally finite Souslin scheme. The key lemma which allows us to do so is the following.

Lemma 3.33. *Let (U_n) be a family of open subsets of a Polish space. There is a family (V_n) of open sets such that*

(i) for every $n \in \mathbb{N}$, $V_n \subseteq U_n$;

(ii) $\bigcup_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} U_n$;

(iii) every $x \in \bigcup_n V_n$ belongs to finitely many V_n's.

Proof. We would like to simply build a partition by letting $V_n = U_n \setminus \bigcup_{i<n} U_i$ but of course V_n may simply not be open. However each U_i is F_σ so we will simply remove more and more of U_i from U_n as n grows so as to satisfy (iii). To be more precise, for each $i \in \mathbb{N}$ we write U_i as

$$U_i = \bigcup_n F_{i,n}$$

where $F_{i,n}$ is an increasing sequence of closed sets. Then let

$$V_n = U_n \setminus \bigcup_{i<n} F_{i,n}.$$

Assertions (i) and (ii) are clearly satisfied, and to see that (iii) holds, let $i \in \mathbb{N}$ and suppose $x \in V_i = \bigcup_n F_{i,n}$. Let $n \in \mathbb{N}$ such that $x \in F_{i,n}$ then by construction x cannot belong to U_m as soon as $m \geqslant n$. □

Let us apply the above lemma a first times to (U_n) and get (V_n) as in the lemma. Then for each n apply the lemma to $(V_n \cap U_{n,m})$ and keep on doing so (apply to $V_s \cap U_{sm}$). We remark that since for every $n \in \mathbb{N}$ we have $Y \subseteq \bigcup_{|s|=n} U_s$, the same is true with (V_s) by an immediate induction. □

3.8 A characterization of the Baire space

Chapter 4

Examples of Polish spaces

4.1 Function spaces

4.1.1 The compact-open topology

We already mentioned that every separable Banach space is a Polish space for the topology induced by its norm. An example of separable Banach space that the reader may have already encountered is the space of continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Our first class of examples of Polish spaces are a generalization of this space: we will show that whenever X is a locally compact Polish space and Y is any Polish space, the space of continuous functions from X to Y has a natural Polish topology.

Definition 4.1. Let X be a locally compact Polish space, let Y be a Polish space. We denote by $\mathcal{C}(X,Y)$ the space of continuous maps from X to Y.

It is equipped with the **compact-open topology**, which is the topology generated by the subbasis consisting of the subsets

$$\{f \in \mathcal{C}(X,Y) : f(K) \subseteq U\}$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

Our main result in this section is that this topology is always Polish. Let us start by understanding better this topology when X is compact Polish.

Recall that if X is compact then by Cor. 1.111 every continuous function $X \to Y$ is d-bounded (meaning that its image has finite diameter) and thus $\mathcal{C}(X,Y)$ is a subspace of the space $\ell^\infty_d(X,Y)$ of all d-bounded functions.

Recall that the space $\ell^\infty_d(X,Y)$ is equipped with the metric d_∞ of **uniform convergence** defined by

$$d^\infty(f,g) = \sup_{x\in X} d(f(x),g(x)).$$

We saw in Prop. 1.80 that such a metric is complete, and we will see that it is actually compatible and complete on $\mathcal{C}(X,Y)$ when X is compact Polish.

Proposition 4.2. *Let X be a compact Polish space and (Y,d) a metric space. Then the compact-open topology on $\mathcal{C}(X,Y)$ is the same as the topology of uniform convergence induced by d_∞.*

Proof. Let us first show that every element of the subbasis of the compact-open topology is d_∞-open: let $K \subseteq X$ compact, $U \subseteq Y$ open, let $O = \{f \in \mathcal{C}(X,Y) : f(K) \subseteq U\}$ and finally let $f_0 \in O$.

Recall that $f_0(K)$ is compact (see Thm. 1.127). Let $\epsilon > 0$ such that all the elements of $f_0(K)$ are at distance at least ϵ from all the elements in the complement of U (such an ϵ exists by virtue of Prop. 1.126). Then all the elements f of the d^∞-ball of radius ϵ around f_0 still satisfy $f(K) \subseteq U$, so that O is d^∞-open as wanted.

Conversely, let us show that every d_∞ open ball is a neighborhood of its center for the compact-open topology. Let $f_0 \in \mathcal{C}(X,Y)$, let $\epsilon > 0$. By compactness the cover $(f_0^{-1}(B(y,\epsilon/5)))_{y\in Y}$ admits a finite subcover $(f_0^{-1}(B(y_i,\epsilon/4)))_{i=1}^n$. For $i = 1,...,n$ let $C_i = f_0^{-1}(B^{\leqslant}(y_i,\epsilon/5))$, then $(C_i)_{i=1}^n$ is a cover of X by closed (hence compact) subsets such that for each i, we have $\operatorname{diam}(f_0(C_i)) < \epsilon/2$.

For each $i \in \{1,...,n\}$, pick $x_i \in C_i$ and let U_i be the open ball of radius $\epsilon/2$ around $f_0(x_i)$. Then by construction the intersection over $i \in \{1,...,n\}$ of the open sets

$$\{f \in \mathcal{C}(X,Y) : f(C_i) \subseteq U_i\}$$

is contained in the ϵ-ball around f_0. Indeed given any $x \in K$, take $i \in \{1,...,n\}$ such that $x \in C_i$, then both $f(x)$ and $f_0(x)$ belong to U_i and thus are at distance at most ϵ. We conclude that $B_{d_\infty}(f_0,\epsilon)$ is a neighborhood of f_0 for the compact-open topology as wanted. □

Observe that the above proposition implies that if we take two compatible metrics d_1 and d_2 on Y, then the restrictions of the metrics d_1^∞ and d_2^∞ to $\mathcal{C}(X,Y)$ are also compatible, which was not clear at all a priori.

We will now see that d^∞ is complete when restricted to $\mathcal{C}(X,Y)$ when X is compact, using the fact that $\mathcal{C}(X,Y)$ then consists of *uniformly* continuous functions.

Definition 4.3. Let (X,d_X) and (Y,d) be metric space, the space $\mathcal{UC}(X,Y)$ is the space of uniformly continuous functions, i.e. of functions $f : X \to Y$ such that for all $\epsilon > 0$, there is $\delta > 0$ such that whenever $x_1, x_2 \in X$ satisfy $d_X(x_1,x_2) < \delta$, we have $d(f(x_1),f(x_2)) < \epsilon$.

Lemma 4.4. *Let (X,d_X) and (Y,d) be metric space. The space $\mathcal{UC}(X,Y)$ is closed in $\ell_d^\infty(X,Y)$.*

Proof. For a function $f \in \ell_d^\infty(X,Y)$, its **modulus of uniform continuity** is the map $\delta(f) : \mathbb{N} \to [0,+\infty[$ defined by

$$\delta(f)(n) = \sup_{d_X(x_1,x_2)<\frac{1}{n}} d(f(x_1),f(x_2)).$$

Observe that since f is bounded, we have $\delta(f) \in \ell^\infty(\mathbb{N},\mathbb{R})$. Moreover it is straightforward to check that $\delta : \ell_d^\infty(X,Y) \to \ell^\infty(\mathbb{N},\mathbb{R})$ is 2-Lipschitz, hence continuous.

Now by definition f is uniformly continuous if and only if $\delta(f)(n) \to 0[n \to +\infty]$, so by the continuity of δ it suffices to show that the following claim holds.

Claim. In $\ell^\infty(\mathbb{N},\mathbb{R})$, the space of sequences which converge to zero is closed.

Proof of claim. We show that its complement is open: if $u_n \not\to 0$, we may find $\epsilon > 0$ and a subsequence $(u_{\varphi(n)})$ such that $|u_{\varphi(n)}| \geqslant \epsilon$ for all $n \in \mathbb{N}$. Then if (v_n) is $\epsilon/2$-close to (u_n) it will satisfy $|v_{\varphi(n)}| \geqslant \epsilon/2$ for all n and hence will not converge to zero. This proves that the space of sequences which do not converge to zero is open as wanted. □

As explained before, since $\mathcal{UC}(X,Y)$ is the inverse image of the space of sequences converging to zero via the continuous map δ, we then have that $\mathcal{UC}(X,Y)$ is closed as wanted. □

Exercise 4.1. Show that $\mathcal{UC}(X,Y)$ is not separable for the topology of uniform convergence as soon as X is non compact. (Hint: Prove this first when $X = \mathbb{N}$ and $Y = \{0,1\}$, both equipped with the discrete metric).

Proposition 4.5. *Let X be a compact space, let (Y,d) be a complete metric space. The the metric d^∞ on $\mathcal{C}(X,Y)$ is complete.*

Proof. Let d_X be a compatible metric on X, then we know that every continuous function $X \to Y$ is uniformly continuous, so $\mathcal{C}(X,Y) = \mathcal{UC}(X,Y)$ by Prop. 1.115. By the two previous lemmas, the latter is closed in the complete metric space $(\ell^\infty_d(X,Y), d^\infty)$ and hence it is a complete metric space for the induced metric by Prop. 1.78. □

Now that we have a complete metric on $\mathcal{C}(X,Y)$ compatible with the compact-open topology when X is compact and Y is Polish, we need to check that the compact-open topology is separable. Let us first do this in the easier case where X is the Cantor space.

Lemma 4.6. *Let Y be a Polish space. Then $\mathcal{C}(2^{\mathbb{N}},Y)$ is separable for the compact-open topology.*

Proof. Let D be a countable dense subset of Y. For each $n \in \mathbb{N}$, consider the countable space D_n of functions taking values in D such that $f(x)$ only depends on the first n bits of x. We will show that the countable set $\bigcup_{n\in\mathbb{N}} D_n$ is dense in $\mathcal{C}(2^{\mathbb{N}},Y)$ by using the compatible metric d^∞ associated to some compatible metric d on Y.

Let $\epsilon > 0$ and let $f \in \mathcal{C}(2^{\mathbb{N}},Y)$. For each $x \in 2^{\mathbb{N}}$ there is a clopen set U_x containing x such that $\operatorname{diam} f(U_x) < \epsilon$, so by compactness we may find a finite subcover $(U_1,...,U_n)$ of $2^{\mathbb{N}}$ by clopen sets such that for each $i \in \{1,...,n\}$, $\operatorname{diam}_d(f(U_i)) < \epsilon$.

Now there is $N \in \mathbb{N}$ such that each U_i is a reunion of cylinder sets of length N. So if we pick for each $s \in 2^n$ some $\tilde{s} \in N_s$ and define a function $f_N \in D_N$ by $f_N(x) = f(\widetilde{x_{\restriction N}})$, we see that $d^\infty(f, f_N) < \epsilon$ as wanted. □

Theorem 4.7. *Let X be a compact Polish space. Then $\mathcal{C}(X,Y)$ is a Polish space for the compact-open topology.*

Proof. As explained before, we already have a compatible complete metric on $\mathcal{C}(X,Y)$: the metric of uniform convergence d^∞ associated to a complete metric d on Y. Indeed by Prop. 4.2 the compact-open topology is compatible with the associated metric of uniform convergence d^∞ and by Prop. 4.5 the latter is complete.

To show that $\mathcal{C}(X,Y)$ is Polish for the compact-open topology, we now only need to show it is separable. To this end, let $\pi : 2^{\mathbb{N}} \to X$ be a continuous surjection as provided by Thm. 3.11. Then every continuous function $f : X \to Y$ lifts to a continuous function $\tilde{f} : 2^{\mathbb{N}} \to Y$ given by $\tilde{f} = f \circ \pi$. Moreover $f \mapsto \tilde{f}$ is easily checked to be an isometry $\mathcal{C}(X,Y) \to \mathcal{C}(2^{\mathbb{N}},Y)$ for their metrics of uniform convergence associated to d. By the previous lemma the metric space $\mathcal{C}(2^{\mathbb{N}},Y)$ is separable, so the isometrically embedded space $\mathcal{C}(X,Y)$ also is by Cor. 1.97. □

Let us now deal with the general case when X is locally compact Polish. Recall that by Thm. **??** we may then write $X = \bigcup_{n\in\mathbb{N}} K_n$ where each K_n is compact and contained in the interior of K_{n+1}.

Proposition 4.8. *Let X be a locally compact non compact Polish space, let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of X such that $X = \bigcup_{n\in\mathbb{N}} K_n$ and each K_n is compact and contained in the interior of K_{n+1}. For each $n \in \mathbb{N}$, let $r_n : \mathcal{C}(X,Y) \to \mathcal{C}(K_n,Y)$ and $\pi_n : \mathcal{C}(X,K_{n+1}) \to \mathcal{C}(K_n,Y)$ be the restriction maps.*

Then the map

$$f \mapsto (r_n(f))_{n\in\mathbb{N}}$$

induces a homeomorphism between $\mathcal{C}(X,Y)$ and $\varprojlim \mathcal{C}(K_n,Y)$.

Proof. First note that r_n is continuous because if $K \subseteq K_n$ is compact and $U \subseteq Y$ is open, then the preimage of the subbasic open set $\{f \in \mathcal{C}(K_n,Y) : f(K) \subseteq U\}$ is simply the open set $\{f \in \mathcal{C}(X,Y) : f(K) \subseteq U\}$. Also note that r_n and the π_n's commute so by the universal property of the projective limit we have a unique continuous map $r : \mathcal{C}(X,Y) \to \varprojlim \mathcal{C}(K_n,Y)$ such that $\pi_n r = r_n$.

The map r is injective since $\bigcup_n K_n = X$, and we must thus show that it is surjective and open. Both actually follow from the fact that each K_n is contained in the interior of K_{n+1}. Indeed, suppose we are given $(f_n) \in \varprojlim \mathcal{C}(K_n,Y)$, i.e. a sequence of continuous functions $f_n : K_n \to Y$ where f_{n+1} extends f_n for every n. We are then forced to defined the map f by $f(x) = f_n(x)$ for all $x \in X$, where n is large enough so that $x \in K_n$. The continuity of X follows from the fact that K_n is contained in the interior of K_{n+1}: indeed we can then deduce from the continuity of f_{n+1} the fact that $\lim_{x\to x_0} f(x) = f(x_0)$ for all $x_0 \in K_n$ and conclude that f is continuous since $\bigcup_n K_n = X$. This shows that r is surjective.

Let us finally show r is open. Consider $K \subseteq X$ compact and $U \subseteq Y$ is open, by compactness and the equality $X = \bigcup_n \mathring{K}_n$ there is n_0 such that $K \subseteq K_{n_0}$. Then it is straightforward to check that

$$r(\{f \in \mathcal{C}(X,Y) : f(K) \subseteq U\}) = \{(f_n) \in \varprojlim \mathcal{C}(K_n,Y) : f_{n_0}(K) \subseteq U\}$$

which yields that the bijection r is open as wanted. □

4.1.2 The topology of pointwise convergence and equicontinuity

Definition 4.9. Let X be a topological space, an **equicontinuity modulus** on X is a family $\mathcal{V} = (V_{x,\epsilon})_{x\in X,\epsilon>0}$ be a family of subsets of X such that for each $x \in X$ and $\epsilon > 0$, $V_{x,\epsilon}$ is an open neighborhood of x.

If (Y,d) is a metric space and $f : X \to Y$, we say that f is $\mathcal{V}$-continuous if for all $x \in X$ and all $\epsilon > 0$ we have

$$\mathrm{diam}_d(f(V_{x,\epsilon})) \leqslant \epsilon.$$

Observe that a function $f : X \to Y$ is continuous if and only if it admits some equicontinuity modulus. A family $(f_i)_{\in I}$ of continuous functions is called **equicontinuous** if there is an equicontinuity modulus $\mathcal{V}$ such that each f_i is $\mathcal{V}$-continuous.

Theorem 4.10. *Let X be a Polish space, let (Y,d) be a complete separable metric space. Let $\mathcal{V} = (V_{x,\epsilon})_{x\in X,\epsilon>0}$ be an equicontinuity modulus. Consider the set $\mathcal{F}_\mathcal{V}$ of functions which are $\mathcal{V}$-continuous and let D be a countable dense subset of X.*

Then the restriction map

$$\Phi : \mathcal{F}_\mathcal{V} \to Y^D$$
$$f \mapsto f_{\restriction D}.$$

is a homeomorphism onto its image, and $\Phi(\mathcal{F}_\mathcal{V})$ is closed in Y^D.

Proof. Let us show that Φ is a homeomorphism onto its image. Since continuous functions are determined by their restriction to a dense subset, the map Φ is injective. It is continuous by definition of the topology of pointwise convergence. To see that Φ^{-1} is continuous, let $x \in X$, let $f \in \mathcal{F}_\mathcal{V}$ and let $\epsilon > 0$. Consider the neighborhood $V_{x,\epsilon/3}$ of x and fix $y \in D \cap V_{x,\epsilon/3}$. Observe that if $d(f(y), g(y) < \epsilon/3$ then by the triangle inequality $d(f(x), g(x)) < \epsilon$. We conclude Φ^{-1} is open.

Let us now show that $\Phi(\mathcal{F}_\mathcal{V})$ is closed in Y^D. Observe that $\Phi(\mathcal{F}_\mathcal{V})$ is contained in the set $\mathcal{G}$ of $f \in Y^D$ such that

$$\forall x \in X, \forall \epsilon > 0, \forall y_1, y_2 \in D \cap V_{x,\epsilon}, d(f(y_1), f(y_2)) \leqslant \epsilon.$$

The set $\mathcal{G}$ is clearly closed, moreover all its element have oscillation zero on X so they are restrictions of continuous functions on X which still have $\mathcal{V}$ as a modulus of oscillation. So $\Phi(\mathcal{F}_\mathcal{V}) = \mathcal{G}$ is closed as wanted. □

Theorem 4.11. *Let X be a topological space, let $\mathcal{V} = (V_{x,\epsilon})$ be an equicontinuity modulus on X and let (Y, d) be a metric space. Then on $\mathcal{F}_\mathcal{V}$ the compact-open topology and the topology of pointwise convergence coincide.*

Proof. By the definition of the compact-open topology and the fact that whenever K is compact the compact-open topology is induced by the metric of uniform convergence, we need to show that whenever K is compact, the topology of pointwise convergence and the topology of uniform convergence coincide on $\mathcal{F}_V$. The topology of uniform convergence clearly contains the topology of pointwise convergence. For the converse, we will show that every neighborhood for the uniform convergence is also a neighborhood fro the pointwise convergence.

Let $f \in \mathcal{F}_V$, and let $\epsilon > 0$. By compactness we find a finite cover $(V_{x_i,\epsilon})_{i=1}^n$ of K. Then by the triangle inequality and equicontinuity we have that $B_{d^\infty}(f, 3\epsilon)$ contains the finite intersection

$$\bigcap_{i=1}^n \{g \in \mathcal{F}_V : d(g(x_i), f(x_i)) < \epsilon\},$$

and is thus a neighborhood of f for the pointwise convergence topology. □

Corollary 4.12 (Ascoli's theorem)**.** *Let $(f_i)_{i \in I}$ be a sequence of continuous functions on a compact Polish space X. Then $(f_i)_{i \in I}$ is relatively compact if and only if it is equicontinuous and for each $x \in X$, the set $\{f_i(x) : i \in I\}$ is relatively compact.*

Proof. Assume $(f_i)_{i \in I}$ is relatively compact. □

4.2 Topologies on hyperspaces of closed subsets

4.2.1 The lower topology on the hyperspace of closed subsets of a Polish space

4.2.2 The Vietoris topology on the hyperspace of compact subsets of a Polish space

4.2.3 The Fell topology on the hyperspace of closed subsets of a locally compact Polish space

4.2.4 The Wijsman topology on the hyperspace of closed subsets of a complete separable metric space

Let (X, d) be a complete separable metric space. The Wijsman topology on $F(X)$ is the smallest topology which makes for every $x \in X$ the map $F \mapsto d(x, F)$ continuous.

Theorem 4.13. *Let (X, d) be a complete separable metric space. Then the Wijsman topology on $F(X)$ is Polish.*

Proof. We will show that Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. Consider the space

Observe that the space of 1-Lipschitz maps $X \to \mathbb{R}$ is closed in $\mathbb{R}^X$. Moreover, the

$$\Phi : F(X) \to \mathbb{R}^X$$

□

4.3 Spaces of structures

Only in the classical case.

4.4 Further examples

Mentions further examples such as space of actions, L^0 spaces (proofs and defs to be given later). Metric structures. Spaces of models for universal theories. Operator algebras. Spaces of operator algebras.

4.5 Exercices

Add the Michael selection theorem.

Chapter 5

Borel sets and functions

5.1 The Borel hierarchy

We have already seen several important classes of subsets in a Polish space such as open, closed, G_δ and F_σ sets. These are part of the fundamental class of Borel subsets, and we will start by recalling its definition.

A **σ-algebra** on a set X is set $\mathcal{A} \subseteq \mathcal{P}(X)$ of subsets of X such that

- $\mathcal{A}$ contains the empty set: $\emptyset \in \mathcal{A}$;
- $\mathcal{A}$ is stable under taking complements: for all $A \in \mathcal{A}$, we also have $X \setminus A \in \mathcal{A}$;
- $\mathcal{A}$ is stable under countable unions: for all $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, one has $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

The set of all subsets $\mathcal{P}(X)$ is a σ-algebra, and any intersection of σ-algebras is a σ-algebra. Given $\mathcal{B} \subseteq \mathcal{P}(X)$, we can thus define the σ-algebra **generated** by $\mathcal{B}$ as the intersection of all the σ-algebras containing $\mathcal{B}$, which by construction is the smallest σ-algebra containing $\mathcal{B}$. Note that this definition is *from above* since it requires us to consider bigger σ-algebras.

Definition 5.1. Let (X, τ) be a topological space. The σ-algebra of **Borel subsets** of X is the σ-algebra generated by the topology of X. It is denoted by $\mathcal{B}(X, \tau)$ or simply $\mathcal{B}(X)$ when the topology is clear from the context.

We will now see how to construct the Borel σ-algebra *from below*, starting from open sets and taking sufficiently many countable unions and complements. We will need a construction by induction on countable ordinals to ensure stability under countable unions in the end.

Definition 5.2. Let X be a topological space. We define by induction on $\xi \in \omega_1^*$ some sets of subsets of X denoted by $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$ as follows:

(a) $\mathbf{\Sigma}^0_1(X)$ is the set of open subsets of X;

(b) For all $\xi \in \omega_1$ with $\xi \geqslant 1$, $\mathbf{\Pi}^0_\xi(X) = \{X \setminus A : A \in \mathbf{\Sigma}^0_\xi(X)\}$;

(c) For all $\xi \in \omega_1$ with $\xi \geqslant 2$, $\mathbf{\Sigma}^0_\xi$ is the set of $A \subseteq X$ that can be written as $A = \bigcup_{n\in\mathbb{N}} A_n$ where for all $n \in \mathbb{N}$,
$$A_n \in \bigcup_{\eta<\xi} \Pi^0_\eta(X).$$

Elements of $\mathbf{\Sigma}^0_\xi(X)$ are called of **additive class** ξ while elements of $\mathbf{\Pi}^0_\xi(X)$ are of **multiplicative class** ξ. Finally, we let

$$\mathbf{\Delta}^0_\xi(X) = \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X)$$

and refer to elements of $\mathbf{\Delta}^0_\xi(X)$ as of **ambiguous class** ξ.

Let us compute the very first steps of this construction. We start by (a) with $\mathbf{\Sigma}^0_1(X)$ the set of open subsets of X, then by (b) we take complements and obtain $\mathbf{\Pi}^0_1(X)$ which is thus the set of closed subsets of X. After that, we take by (c) countable unions and get $\mathbf{\Sigma}^0_2(X)$ the set of F_σ subsets of X and finally by (b) again $\mathbf{\Pi}^0_2(X)$ is the set of complements of F_σ subsets of X, also known as G_δ subsets of X. Also note that by definition a subset of X is of ambiguous class 1 (belongs to $\mathbf{\Delta}^0_1(X)$) if and only if it is clopen.

One often denotes by $G(X) = \mathbf{\Sigma}^0_1(X)$ the set of open subsets of X and by $F(X) = \mathbf{\Pi}^0_1(X)$ the set of closed subsets of X. The notations F_σ and G_δ can be generalised: given a set of subsets $\mathcal{A}$, we denote by $\mathcal{A}_\sigma$ the set of countable unions of elements of $\mathcal{A}$, and by $\mathcal{A}_\delta$ the set of countable intersections of elements of $\mathcal{A}$. As an example, axiom (c) can be rewritten as

$$\mathbf{\Sigma}^0_\xi(X) = \left(\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(X)\right)_\sigma .$$

We will also drop the reference to X when using the above notations as adjectives, which we were already doing when talking about F_σ or G_δ subsets of X. For instance, when we write "let A be a $\mathbf{\Pi}^0_3$ subset of X", we mean "let $A \in \mathbf{\Pi}^0_3(X)$".

Remark 5.3. One reads $\mathbf{\Pi}^0_3$ as Pi-zero-three (the order is important because when we will go beyond the Borel hierarchy, the upper index will change !).

Lemma 5.4. *Let X be a Polish space. For all $\xi, \eta \in \omega_1$ such that $\eta \leqslant \xi$, we have*

$$\mathbf{\Sigma}^0_\eta(X) \subseteq \mathbf{\Sigma}^0_\xi(X) \text{ and } \mathbf{\Pi}^0_\eta(X) \subseteq \mathbf{\Pi}^0_\xi(X).$$

Proof. First note that $\mathbf{\Sigma}^0_\eta(X) \subseteq \mathbf{\Sigma}^0_\xi(X)$ if and only if $\mathbf{\Pi}^0_\eta(X) \subseteq \mathbf{\Pi}^0_\xi(X)$ since we can go from one to the other by taking complements, which is involutive.

Let us then show $\mathbf{\Sigma}^0_\eta(X) \subseteq \mathbf{\Sigma}^0_\xi(X)$ whenever $\eta \leqslant \xi$. Note that if both $\xi \geqslant 2$ and $\eta \geqslant 2$ then by (c) we have that $\mathbf{\Sigma}^0_\eta(X) \subseteq \mathbf{\Sigma}^0_\xi(X)$ since the latter is the set of countable unions of more sets.

We now only have to show that $\mathbf{\Sigma}^0_1(X) \subseteq \mathbf{\Sigma}^0_2(X)$. This inclusion means that every open set is F_σ, a fact that we have already observed (see Rmk. **??**). □

Remark 5.5. Duality here ? Then obtain Pi as countable intersection of Sig.

Note that by the above lemma we have $\mathbf{\Sigma}^0_{\xi+1} = (\mathbf{\Pi}^0_\xi)_\sigma$ (DEFINE NOTATIONS). More generally, the following is true.

Lemma 5.6. *Suppose (η_n) is a countable family of ordinals and let ξ be their strict supremum. Then*

$$\mathbf{\Sigma}^0_\xi = (\bigcup_n \mathbf{\Pi}^0_{\eta_n})_\sigma \text{ and } \mathbf{\Pi}^0_\xi = (\bigcup_n \mathbf{\Sigma}^0_{\eta_n})_\delta$$

We then have the following basic stability properties for the classes of Borel subsets $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$, where X and Y denote Polish spaces.

- Stability under continuous preimages: if Γ is either $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$ for some $\xi \in \omega_1$, then for every continuous map $f : X \to Y$ and every $A \in \Gamma(Y)$ we have $f^{-1}(A) \in \Gamma(Y)$.

- Stability under finite unions or intersections: if Γ is either $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$ for some $\xi \in \omega_1$ then for all $A, B \in \Gamma(X)$ we have $A \cap B \in \Gamma(X)$ and $A \cup B \in \Gamma(X)$.

- $\mathbf{\Sigma}^0_\xi$ is actually stable under *countable* unions: for all $\xi \in \omega_1$ and for all $(A_n) \in (\mathbf{\Sigma}^0_\xi(X))^{\mathbb{N}}$ we have
$$\bigcup_{n\in\mathbb{N}} A_n \in \mathbf{\Sigma}^0_\xi(X).$$

- $\mathbf{\Pi}^0_\xi$ is stable under countable intersections: for all $\xi \in \omega_1$ and for all $(A_n) \in (\mathbf{\Pi}^0_\xi(X))^{\mathbb{N}}$ we have
$$\bigcap_{n\in\mathbb{N}} A_n \in \mathbf{\Pi}^0_\xi(X).$$

Exercise 5.1. Establish the above four stability properties of the classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.

5.2 Decompositions as disjoint unions and applications

The disjointness trick allows us to write every reunion of Borel sets $\bigcup_n A_n$ as a reunion of disjoint Borel sets $\bigsqcup_n A'_n$ with $A'_n \subseteq A_n$. Let us apply this trick to the definition of the class $\mathbf{\Sigma}^0_\xi$ for $\xi > 1$.

Proposition 5.7. *Let $\xi > 1$. Then every element of $\mathbf{\Sigma}^0_\xi$ can be written as a countable disjoint union of Δ^0_ξ sets.*

Proof. Let $A = \bigcup_n B_n$ where each B_n is in some $\mathbf{\Pi}^0_\eta$ for some $\eta < \xi$. For every $n \in \mathbb{N}$ let $A'_n := A_n \setminus (A_0 \cup \cdots \cup A_{n-1})$. Then $A'_n \in \Delta^0_\xi$ and $A = \bigsqcup A'_n$. □

Proposition 5.8. *Let $\xi \geqslant 3$. Then every element of $\mathbf{\Sigma}^0_\xi$ can be written as a countable disjoint union of $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta$ sets.*

Proof. By Lemma 5.4 every element of $\mathbf{\Sigma}^0_\xi$ can be written as a countable union of elements of $\bigcup_{1<\eta<\xi} \mathbf{\Pi}^0_\eta$.

So write $A = \bigcup_n A_n$ in such a way. For each n the set $X \setminus A_0 \cup \cdot \cup A_{n-1}$ is in $\mathbf{\Sigma}^0_\eta$ for some $1 < \eta < \xi$. So
$$X \setminus (A_0 \cup \cdots \cup A_{n-1}) = \bigsqcup_m B_{n,m}$$
where $B_{n,m} \in \Delta^0_\eta \subseteq \mathbf{\Pi}^0_\eta$. Now
$$A = \bigsqcup_n A_n \cap (X \setminus (A_0 \cup \cdot \cup A_{n-1})) = \bigsqcup_n \left(A_n \cap \bigsqcup_m B_{n,m} \right) = \bigsqcup_{n,m} A_n \cap B_{n,m}.$$
Now for all n, m we have $A_n \cap B_{n,m} \in \mathbf{\Pi}^0_\eta$ for some $\eta < \xi$ as wanted. □

[Source: Kuratowski, Topologie I p. 254 C'est du à Lusin, cf. ref. therein].

We have a weaker result in the case $\xi = 2$. Let us start by a lemma on particular $\mathbf{\Sigma}^0_2$ sets, namely open sets.

Lemma 5.9. *In a metrizable space, every open set can be written as a reunion of a countable point-finite family of closed sets.*

Proof. Let d be a compatible metric and let U be open. For every $n \in \mathbb{N}$, consider the set

$$F_n := \{x \in X : d(x, X \setminus U) \in [1/n+1, 1/n]\}.$$

Each F_n is closed and we have $U = \{x \in X : d(x, X \setminus U) > 0\} = \bigcup_{n\in\mathbb{N}} F_n$. By construction each $x \in X$ can belong to at most two distinct F_n's (with consecutive indices) so we have the desired result. □

Proposition 5.10. *In a metrisable space, every F_σ set can be written as a reunion of a countable point-finite family of closed sets.*

Proof. Let $A \bigcup_n F_n$, for each n consider the open set set $U_n := X \setminus (F_0 \cup \cdots \cup F_{n-1})$. By the previous lemma we may write it as $U_n = \bigcup_m F_{n,m}$ where $(F_{n,m})$ is a point-finite family of closed sets.

Now $A = \bigsqcup_n (F_n \cap U_n) = \bigcup_{n,m} F_n \cap F_{n,m}$ is written as the reunion of a countable point-finite family of closed sets. □

Theorem 5.11. *Let $\xi \geqslant 1$. Then $\exists^\infty \mathbf{\Pi}^0_\xi = \mathbf{\Pi}^0_{\xi+2}$.*

Proof. The inclusion $\exists^\infty \mathbf{\Pi}^0_\xi = \mathbf{\Pi}^0_{\xi+2}$ is an exercise. For the other way around, suppose first $\xi \geqslant 2$ and let $A \in \mathbf{\Pi}^0_{\xi+2}$. Then write A as a decreasing intersection of $\mathbf{\Sigma}^0_{\xi+1}$ sets A_n. By the previous proposition, each of these may be written as

$$A_n = \bigsqcup B_{n,m}$$

where $B_{n,m}$ is in $\mathbf{\Pi}^0_\xi$. We now claim that $A = \exists^\infty B_{n,m}$. The inclusion from left to right is clear, and for the other way around note that if $x \in \exists^\infty B_{n,m}$ since for each n there is at most one m such that $x \in B_{n,m}$ there must be infinitely many n such that x belongs to some $B_{n,m}$ hence to A_n, and since the A_n are decreasing this means $x \in \bigcap A_n$.

In the case $\xi = 1$ the same argument works with Prop. 5.10 instead. □

Proposition 5.12. *Let X be a Polish space, let (A_n) be a countable family of elements of $\mathbf{\Sigma}^0_\xi(X)$ for some $\xi \geqslant 2$. Then there are pairwise disjoint $A'_n \subseteq A_n$ with $A'_n \in \mathbf{\Sigma}^0_\xi(X)$ and*

$$\bigsqcup_{n\in\mathbb{N}} A'_n = \bigcup_{n\in\mathbb{N}} A_n.$$

Proof. For each $n \in \mathbb{N}$, write $A_n = \bigcup_{m\in\mathbb{N}} B_{n,m}$ with each $B_{n,m} \in \mathbf{\Pi}^0_\eta$ for some $\eta < \xi$, then $\bigcup_n A_n = \bigcup_{n,m} B_{n,m}$. We will now use the disjointness trick but with $\mathbb{N}^2$ as index set instead of $\mathbb{N}$.

Let $<$ be an order on $\mathbb{N}^2$ isomorphic to the usual well-order on $\mathbb{N}$. For each $(n,m) \in \mathbb{N}^2$, let

$$B'_{n,m} = B_{n,m} \setminus \left(\bigcup_{(k,l)<(n,m)} B_{k,l} \right)$$

Since the set of predecessors of each (n,m) is finite, we have $\bigcup_{(k,l)<(n,m)} B_{k,l} \in \mathbf{\Pi}^0_\xi$ and hence $B'_{n,m} \in \mathbf{\Sigma}^0_\xi$. By construction the family $(B'_{n,m})_{n,m\in\mathbb{N}}$ is disjoint and has the same reunion as $(B_{n,m})_{n,m\in\mathbb{N}}$. Let $A'_n = \bigsqcup_m B'_{n,m}$, since $\mathbf{\Sigma}^0_\xi$ is stable under countable unions we have $A'_n \in \mathbf{\Sigma}^0_\xi$ and

$$\bigsqcup_n A'_n = \bigsqcup_{n,m} B'_{n,m} = \bigcup_{n,m} B_{n,m} = \bigcup_n A_n$$

as wanted. □

5.3 Structural properties

5.4 Γ-complete subsets

Given a Borel subset $A \subseteq X$ of a Polish space, we would like to determine the exact Borel complexity of A, i.e. the smallest ordinals ξ and ξ' such that $A \in \mathbf{\Sigma}^0_\xi(X)$ and $A \in \mathbf{\Pi}^0_{\xi'}(X)$. This could for instance allow us to distinguish A from another Borel set B. Let us first compare the complexities of two sets A and B by trying to compute one from the other through a continuous map.

Definition 5.13. Let X and Y be Polish spaces, let $A \subseteq X$ and $B \subseteq Y$. Say that A **continuously reduces** to B if there is a continuous map $f : X \to Y$ such that for all $x \in X$ the condition $x \in A$ is equivalent to $f(x) \in B$, i.e.

$$A = f^{-1}(B).$$

This is written as $(A \subseteq X) \leqslant_c (B \subseteq Y)$, and often the ambient sets X and Y are clear from the context so we simply write $A \leqslant_c B$.

If $A \leqslant_c B$ we think of A as "simpler" than B. Indeed if we need to know wether $x \in A$ we can do this by checking wether $f(x) \in B$.

Exercise 5.2. Prove that $\leqslant_c$ defines a preorder on inclusions $A \subseteq X$. Show that if X, Y, Z are Polish spaces with $X \subseteq Y$ and $(A \subseteq X) \leqslant_c (B \subseteq Z)$ then $(A \subseteq Y) \leqslant_c (B \subseteq Z)$. Does the converse hold ?

The fact that the classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ are stable under countinuous preimages (Exercise **??**) means that $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ form initial segments for $\leqslant_c$.

We would like to define $\mathbf{\Gamma}$-hard subsets $B \subseteq Y$ as subsets more complicated than *any* element of $\mathbf{\Gamma}(X)$ for *every* Polish space X, and $\mathbf{\Gamma}$-complete subsets as those who moreover belong to $\mathbf{\Gamma}(Y)$.

Note however that there might already be no non-constant continuous maps $X \to Y$, for instance when X is connected and Y is totally disconnected. So the complexity of X has to be as low as possible if we want to have a chance for $\mathbf{\Gamma}$-hard or complete subsets to exist. We will thus take X to be always equal to the Baire space, which as we saw surjects continuously onto any Polish space.

Definition 5.14. Let $\mathbf{\Gamma}$ be a class of subsets of Polish spaces, and let X be a Polish space. An inclusion $A \subseteq X$ is called **$\mathbf{\Gamma}$-hard** if for every every $B \in \mathbf{\Gamma}(\mathbb{N}^\mathbb{N})$ we have $B \leqslant_c A$. $A \subseteq X$ is **$\mathbf{\Gamma}$-complete** if moreover $A \in \mathbf{\Gamma}(X)$.

When X is clear from the context we will simply say that A is $\mathbf{\Gamma}$-hard (resp. $\mathbf{\Gamma}$-complete).

Exercise 5.3. Prove that $A \subseteq X$ is Γ-hard (resp. Γ-complete) if and only if $X \setminus A$ is $\check{\Gamma}$-hard (resp. $\check{\Gamma}$-complete).

The previous definitions are not standard, usually one replaces the Baire space $\mathbb{N}^\mathbb{N}$ by any zero-dimensional Polish space Y. This difference does not matter for the classes of subsets we are interested in as the following exercise shows.

Exercise 5.4. Let X be a Polish space and $A \subseteq X$, and suppose that $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\xi$ or $\Gamma = \mathbf{\Pi}^0_\xi$ for some $\xi \in \omega_1^*$. Show that A is $\mathbf{\Gamma}$-hard if and only if for every zero-dimensional Polish space Y and every $B \in \mathbf{\Gamma}(Y)$ we have $B \leqslant_c A$.

Let us now check that $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$-complete sets are as complicated as possible.

Proposition 5.15. *Let $A \subseteq X$ be a $\mathbf{\Sigma}^0_\xi$-complete (resp. $\mathbf{\Pi}^0_\xi$-complete) subset. Then $A \notin \mathbf{\Pi}^0_\xi(X)$ (resp. $A \notin \mathbf{\Sigma}^0_\xi(X)$).*

Proof. Since the classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ are dual of each other, it suffices to prove the statement about $\mathbf{\Sigma}^0_\xi$-complete sets (by Exercise 5.4). Suppose A is $\mathbf{\Sigma}^0_\xi$-complete. Let $B \subseteq \mathbb{N}^\mathbb{N}$ be a $\mathbf{\Sigma}^0_\xi$ non $\mathbf{\Pi}^0_\xi$ subset, whose existence is guaranteed by Cor. **??**. Then since A is $\mathbf{\Sigma}^0_\xi$-hard we have $B \leqslant_c A$. So A cannot be $\mathbf{\Pi}^0_\xi$ because otherwise B would also be by Exercise **??**. □

The most complicated elements of $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$ which we have built so far were universal parametrizations and these are indeed complete.

Proposition 5.16. *Let X be a Polish space and $\mathbf{\Gamma}$ be a class of subsets of Polish spaces. Then every X-parametrization of $\mathbf{\Gamma}(\mathbb{N}^\mathbb{N})$ is $\mathbf{\Gamma}$-hard and every universal X-parametrization of $\mathbf{\Gamma}(\mathbb{N}^\mathbb{N})$ is $\mathbf{\Gamma}$-complete.*

Proof. Suppose $A \subseteq X \times \mathbb{N}^\mathbb{N}$ is an X-parametrization of $\mathbf{\Gamma}(\mathbb{N}^\mathbb{N})$ and let $B \in \mathbf{\Gamma}(\mathbb{N}^\mathbb{N})$. Since A parametrizes $\mathbf{\Gamma}(\mathbb{N}^\mathbb{N})$ we find $x \in X$ such that $B = A_x$. Then $B \leqslant_c A$ via $y \mapsto (x, y)$. We conclude that A is $\mathbf{\Gamma}$-hard. If A was moreover universal it follows from the definitions that A is $\mathbf{\Gamma}$-complete. □

Since X-universal parametrizations for $\mathbf{\Sigma}^0_\xi(\mathbb{N}^\mathbb{N})$ or $\mathbf{\Pi}^0_\xi(\mathbb{N}^\mathbb{N})$ exist as soon as X is uncountable Polish, we conclude that for every $\xi \in \omega_1$, there is a $\mathbf{\Sigma}^0_\xi$-complete set as well as a $\mathbf{\Pi}^0_\xi$-complete set. We will now build more trackable examples of such sets using the following simple but fundamental device.

Proposition 5.17. *Consider a class $\mathbf{\Gamma}$ of subsets of Polish spaces. Let $A \subseteq X$ and $B \subseteq Y$ where X and Y are Polish spaces and suppose $A \leqslant_c B$. If A is $\mathbf{\Gamma}$-hard then B also is.*

Proof. Let $C \in \mathbf{\Gamma}(\mathbb{N}^\mathbb{N})$. Then $C \leqslant_c A$ because A is $\mathbf{\Gamma}$-hard. Hence by transitivity $C \leqslant_c B$ and we conclude that B is $\mathbf{\Gamma}$-hard. □

This proposition provides a nice recipe to prove that a subset $B \subseteq Y$ is Γ-complete: first show that $B \in \Gamma(Y)$ and then find a Γ-hard subset $A \subseteq X$ such that $A \leqslant_c B$ (when Γ is stable under continuous preimages the subset A actually had to be Γ-*complete*).

We will now climb up the Borel hierarchy so as to build some easier concrete examples of $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$-complete sets for $\xi \leqslant 3$.

Proposition 5.18. *Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$ as defined in Exercise **??**. Then $\mathbb{N} \subseteq \overline{\mathbb{N}}$ is $\mathbf{\Sigma}^0_1$-complete.*

Proof. Let U be a $\mathbf{\Sigma}^0_1$ (i.e. open) subset of $\mathbb{N}^\mathbb{N}$. For $x \in U$ let $n(x)$ be the smallest of all integer $n \in \mathbb{N}$ such that $N_{x\restriction n} \subseteq U$. Then define $f : \mathbb{N}^\mathbb{N} \to \overline{\mathbb{N}}$ by

$$f(x) = \begin{cases} n(x) & \text{if } x \in U, \\ \infty & \text{else.} \end{cases}$$

By construction $U = f^{-1}(\mathbb{N})$ and one easily checks that f is continuous so that $U \leqslant_c \mathbb{N}$. So $\mathbb{N} \subseteq \overline{\mathbb{N}}$ is $\mathbf{\Sigma}^0_1$-complete. □

Exercise 5.5. Show that in any Polish space X, an open subset is $\mathbf{\Sigma}^0_1$-complete if and only if it is not closed. (*Hint:* use the two preceding propositions)

We now explain how to build a $\mathbf{\Pi}^0_{\xi+1}$-complete set from a $\mathbf{\Sigma}^0_\xi$ one.

Proposition 5.19. *Let X be a Polish space, suppose $A \subseteq X$ is $\mathbf{\Sigma}^0_\xi$-complete. Then $A^{\mathbb{N}} \subseteq X^{\mathbb{N}}$ is $\mathbf{\Pi}^0_{\xi+1}$-complete.*

Proof. First note that $A^{\mathbb{N}} = \bigcap_{n\in\mathbb{N}} \pi_n^{-1}(A)$ where π_n is the projection on the n-th coordinate so that $A^{\mathbb{N}}$ is in $\mathbf{\Pi}^0_{\xi+1}(X^{\mathbb{N}})$.

Now suppose $B \in \mathbf{\Pi}^0_{\xi+1}(\mathbb{N}^{\mathbb{N}})$. By exercise **??** there is a countable family (B_n) of elements of $\mathbf{\Sigma}^0_\xi(\mathbb{N}^{\mathbb{N}})$ such that $B = \bigcap_{n\in\mathbb{N}} B_n$. We will now use the intersection-to-product trick: the map

$$\begin{aligned} \Phi : \mathbb{N}^{\mathbb{N}} &\to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \\ x &\mapsto (x, x, ...) \end{aligned}$$

is continuous and $B = \Phi^{-1}(\prod_{n\in\mathbb{N}} B_n)$ so $B \leqslant_c \prod_{n\in\mathbb{N}} B_n$. We thus only need to show that $\prod_{n\in\mathbb{N}} B_n \leqslant_c A^{\mathbb{N}}$. But since A is $\mathbf{\Sigma}^0_\xi$ complete we have for every $n \in \mathbb{N}$ a continuous map $f_n : \mathbb{N}^{\mathbb{N}} \to X$ such that $B_n = f_n^{-1}(A)$, and then the map $(f_n)_{n\in\mathbb{N}} : \prod_{n\in\mathbb{N}} B_n \to A^{\mathbb{N}}$ witnesses that $\prod_{n\in\mathbb{N}} B_n \leqslant_c A^{\mathbb{N}}$. □

Exercise 5.6. Show more generally that given a countable ordinal ξ and a family (η_n) of ordinals whose strict supremum is equal to ξ, if for every $n \in \mathbb{N}$ the inclusion $A_n \subseteq X_n$ is $\Sigma^0_{\eta_n}$-complete then $\prod_{n\in\mathbb{N}} A_n \subseteq \prod_{n\in\mathbb{N}} X_n$ is Π^0_ξ-complete.

Deduce that $\{0,1\}^{\mathbb{N}}$ contains $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$-complete subsets for every $\xi \in \omega_1^*$. Conclude that the same is true of any uncountable Polish space.

As an immediate consequence of Prop. 5.19 and Prop. 5.18 we get the following result.

Corollary 5.20. *The subset $\mathbb{N}^{\mathbb{N}} \subseteq (\overline{\mathbb{N}})^{\mathbb{N}}$ is $\mathbf{\Pi}^0_2$-complete.*

Deduce that finite subsets of $\mathbb{N}$ are also (prolongement du truc $\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$). So sequences of finite subsets are $\mathbf{\Pi}^0_3$-complete. Use this to show that sequences of integers tending to infinity is also. (show that increasing sequences of finite subsets with union equal to $\mathbb{N}$ correspond to sequences of integers tending to infinity via $u_n \mapsto (A_n := \{k : u_k \leqslant n\})$

Exercise 5.7. Set of non atomic proba is G_δ-complete (associate to zero the proba δ_0 and to one the proba $1/2(\delta_0 + \delta_1)$ to a sequence of zeros and ones associate a product probability measure on $2^{\mathbb{N}}$ as product of the corresponding measures). Deduce that the set of completely atomic is $\mathbf{\Pi}^0_3$-complete (use the sum of the $1/2^n\mu_n$ where μ_n constructed as before).

5.5 The Baire hierarchy of Borel functions

Chapter 6

Standard Borel spaces

This does NOT require injective images of Borel sets being Borel because for Cantor-Bendixon we know already that the maps are homeomorphisms onto their images.

6.1 Turning Borel sets into clopen sets

In this section we will see how to make a Borel subset clopen without changing the Borel σ-algebra. Let us first see how to do this for open subsets.

Lemma 6.1. *Let U be an open subset on a Polish space (X, τ). Then there is a Polish topology τ' on X such that $\tau \subseteq \tau'$, $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau')$ and U is τ'-clopen*

Proof. Both U and $X \setminus U$ are Polish spaces for the induced topology. We let τ' be the disjoint union topology on $U \sqcup (X \setminus U)$, which is Polish by Proposition 2.14. By definition the τ'-open sets are the sets of the form $V \cap U \sqcup (W \cap X \setminus U)$ where V and W are open in X, so $\tau \subseteq \tau'$ and τ'-open sets belong to $\mathcal{B}(X, \tau)$, which implies $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau')$. □

Next, we need a lemma which will help us in proving that the set of Borel subsets which can be made clopen is stable under countable reunion.

Lemma 6.2. *Let (X, τ) be a Polish space. For each $n \in \mathbb{N}$, let τ_n be a topology on X containing τ such that $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau_n)$. Then the topology τ_∞ generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is Polish and satisfies $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau_\infty)$.*

Proof. Consider the following injective map

$$\begin{aligned} \Phi : (X, \tau_\infty) &\to \prod_{n \in \mathbb{N}} (X, \tau_n) \\ x &\mapsto (x, x, \ldots). \end{aligned}$$

Then Φ is a homeomorphism onto its image by Proposition 1.74.

Observe that $\Phi(X)$ is the set of constant maps $\mathbb{N} \to X$, and hence closed in $\prod_{n \in \mathbb{N}} (X, \tau_n)$ because it is closed in $\prod_{n \in \mathbb{N}} (X, \tau)$ (Exercise 1.15) and each τ_n contains τ. We deduce that $\Phi(X)$ is Polish for the topology induced by $\prod_{n \in \mathbb{N}} (X, \tau_n)$, so since Φ is a homeomorphism onto its image the topology τ_∞ is Polish as well.

Finally, τ_∞ is second-countable and generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ so each of its elements is a *countable* reunion of finite intersections of elements of $\bigcup_{n \in \mathbb{N}} \tau_n$. Since for each n we have $\tau_n \subseteq \mathcal{B}(X, \tau)$ this yields $\tau_\infty \subseteq \mathcal{B}(X, \tau)$. We moreover clearly have $\tau \subseteq \tau_\infty$, so $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau_\infty)$. □

Theorem 6.3. *Let (X, τ) be a Polish space. Let $B \subseteq X$ be a Borel subset. Then there is a Polish topology τ' on X containing τ such that B is τ'-clopen, $\tau \subseteq \tau'$ and $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau')$.*

Proof. Let $\mathcal{A}$ be the set of all Borel subsets $B \subseteq X$ satisfying the conclusion of the theorem. We will show that $\mathcal{A}$ is a σ-algebra which contains τ-open sets, from which the desired equality $\mathcal{A} = \mathcal{B}(X, \tau)$ follows by the definition of $\mathcal{B}(X, \tau)$.

Observe that Lemma 6.1 precisely says that $\mathcal{A}$ contains τ-open sets. Moreover $\mathcal{A}$ is stable under complement because the complement of a clopen subset is clopen. We now only need to show that $\mathcal{A}$ is stable under countable reunions.

Let $(A_n)_{n\in\mathbb{N}}$ be a countable family of elements of $\mathcal{A}$, then for each $n \in \mathbb{N}$ we fix a Polish topology τ_n containing τ such that $\mathcal{B}(X, \tau_n) = \mathcal{B}(X, \tau)$ and A_n is τ-clopen. Let τ_∞ be the topology generated by $\bigcup_{n\in\mathbb{N}} \tau_n$. By Lemma 6.2, the topology τ_∞ is Polish and $\mathcal{B}(X, \tau_\infty) = \mathcal{B}(X, \tau)$.

Now A is τ_∞-open as it is the reunion of the τ_∞ open sets A_n. We may thus use Lemma 6.1 so as to get τ' containing τ_∞ with $\mathcal{B}(X, \tau_\infty) = \mathcal{B}(X, \tau')$ and A τ'-clopen. Then τ' witnesses that $A \in \mathcal{A}$ so $\mathcal{A}$ is indeed a σ-algebra. As explained at the beginning, this ends the proof. □

This theorem has many structural consequences. The first is that Borel sets satisfy the a stronger topological form of the continuum hypothesis.

Theorem 6.4. *Let (X, τ) be a Polish space, let $B \subseteq X$ be an uncountable Borel subset. Then B contains a closed subset of X which is homeomorphic to the Cantor space.*

Proof. Let τ' be a Polish topology containing τ such that $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau')$ and B is τ'-clopen. Then B is Polish for the topology induced by τ', and since it is uncountable Corollary 3.10 provides us a continuous injective map $f : 2^{\mathbb{N}} \to (B, \tau')$. In particular f is continuous injective as a map $2^{\mathbb{N}} \to (X, \tau)$, and since $2^{\mathbb{N}}$ is compact we conclude that f is a homeomorphism onto its image and that $f(2^{\mathbb{N}})$ is τ-closed. □

Another consequence is that we can turn Borel functions into continuous ones.

Theorem 6.5. *Let (X, τ) be a Polish space, let Y be a second-countable topological space and let $f : X \to Y$ be a Borel map. Then there is a Polish topology τ' on X which contains τ such that $f : (X, \tau') \to Y$ is continuous and $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau')$.*

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be a basis for the topology of Y. Since f is Borel each $f^{-1}(V_n)$ is Borel, so by Theorem 6.3 for each $n \in \mathbb{N}$ there is a Polish topology τ_n on X containing τ such that $f^{-1}(V_n)$ is τ_n-clopen and $\mathcal{B}(X, \tau_n) = \mathcal{B}(X, \tau)$. By Lemma 6.2 the topology τ' generated by $\bigcup_{n\in\mathbb{N}} \tau_n$ is stilll Polish, contains τ and satisfies $\mathcal{B}(X, \tau) = \mathcal{B}(X, \tau')$. Now for each $n \in \mathbb{N}$ the set $f^{-1}(V_n)$ is τ'-open, so since $(V_n)_{n\in\mathbb{N}}$ is a basis for the topology of Y we conclude $f : (X, \tau') \to Y$ is continuous as wanted. □

The following consequence will prove useful in the next section.

Proposition 6.6. *Let X be a Polish space, let $B \subseteq X$ be a Borel subset. Then there is a a closed subset F of $\mathbb{N}^{\mathbb{N}}$ and a continuous injective map $f : F \to X$ such that $B = f(\mathbb{N}^{\mathbb{N}})$. Moreover f^{-1} is Borel.*

Proof. Denote by τ the topology of X Let τ' be a Polish topology containing τ such that $\mathcal{B}(X,\tau) = \mathcal{B}(X,\tau')$ and B is τ'-clopen. Then B is Polish for the topology induced by τ' so by Theorem 3.30 we find a closed subset F of $\mathbb{N}^\mathbb{N}$ and a continuous bijective map $f : \mathbb{N}^\mathbb{N} \to (B,\tau')$ whose inverse is τ'-Baire class 1 (hence Borel). Since $\tau \subseteq \tau'$ the map f is also τ-continuous, and since $\mathcal{B}(X,\tau) = \mathcal{B}(X,\tau')$ we conclude f^{-1} is also Borel for the topology τ. □

Our final application of Theorem 6.3 says that Borel sets are *analytic* (see Chap. 7).

Proposition 6.7. *Let X be a Polish space, let $B \subseteq X$ be a Borel subset. Then there is a continuous map $f : \mathbb{N}^\mathbb{N} \to X$ such that $B = f(\mathbb{N}^\mathbb{N})$.*

Proof. Compose the continuous map $f : F \to B$ from the previous proposition with a continuous surjection $g : \mathbb{N}^\mathbb{N} \to F$ whose existence is guaranteed by Prop. **??** □

6.2 Classification of standard Borel spaces

6.3 Operations on standard Borel spaces

The following lemma will be upgraded in the next chapter.

Lemma 6.8. *Let X be a Polish space and Y a Hausdorff second-countable topological space. If $f : X \to Y$ is Borel, then the graph of f is Borel.*

Proof. Since Y is Hausdorff, the diagonal $\Delta_Y = \{(y,y) : y \in Y\}$ is closed in Y^2, in particular it is Borel. The map $\Phi : (x,y) \mapsto (f(x),y)$ is Borel, and the graph of f is equal to $\Phi^{-1}(\Delta_Y)$ hence Borel. □

6.4 The Effros space of a Polish space

We will now define an important example of standard Borel space, namely the space of all closed subsets of a given Polish space X, also called the Effros space of X.

Definition 6.9. Let X be a Polish space. The **Effros** space of X is the space $F(X)$ of all closed subsets of X equipped with the σ-algebra generated by sets of the form

$$\mathcal{F}_U := \{F \in F(X) : F \cap U \neq \emptyset\}$$

where U ranges over open subsets of X. This σ-algebra will be called the Effros σ-algebra.

Theorem 6.10. *The Effros space of a Polish space is a standard Borel space.*

Proof. We will show that the Effros space is standard by showing that it is measurably isomorphic to a Borel subspace of the Cantor space $2^\mathbb{N}$.

Let (U_n) be a countable basis of the topology of X consisting of nonempty open sets. Let us first note that the Effros σ-algebra is generated by sets of the form

$$\mathcal{F}_{U_n} = \{F \in F(X) : F \cap U_n \neq \emptyset\}$$

for $n \in \mathbb{N}$. Indeed the σ-algebra generated by such sets is contained in the Effros σ-algebra, and conversely if U is open we write $U = \bigcup_k U_{n_k}$ and thus have $\mathcal{F}_U = \bigcup_k \mathcal{F}_{U_k}$.

We have a natural way of coding a closed subset by noting which U_n it intersects: this defines a map $\Phi : F(X) \to 2^{\mathbb{N}}$ by $\Phi(F) = \{n \in \mathbb{N} : F \cap U_n \neq \emptyset\}$ where we view $2^{\mathbb{N}}$ as the set of subsets of $\mathbb{N}$. Since the complement of a closed set is open and (U_n) is a basis of the topology, for all $F \in \mathcal{F}(X)$ we have

$$X \setminus F = \bigcup_{n \in \mathbb{N} \setminus \Phi(F)} U_n.$$

In particular the map Φ is injective. To see that it is a measurable isomorphism onto its image, simply note that

$$\Phi(\mathcal{F}_{U_n}) = \{A \subseteq \mathbb{N} : n \in A\} \cap \Phi(F(X))$$

and that sets of the form $\{A \subseteq \mathbb{N} : n \in A\}$ for $n \in \mathbb{N}$ generate the Borel σ-algebra of the Cantor space while sets of the form $\mathcal{F}_{U_n}$ for $n \in \mathbb{N}$ generate the Effros σ-algebra as we explained at the beginning of the proof.

To complete the proof, we now need to show the image of Φ is Borel. Let d be a compatible complete metric on X. Clearly $\Phi(F(X))$ is contained in the following two sets

$$\mathcal{B}_1 = \{A \subseteq \mathbb{N} : \forall n, m \in \mathbb{N}, U_n \subseteq U_m \Rightarrow (n \in A \Rightarrow m \in A)\} \text{ and}$$
$$\mathcal{B}_2 = \{A \subseteq \mathbb{N} : \forall n \in A, \forall \epsilon \in \mathbb{Q}^{>0}, \exists m \in A, \overline{U_m} \subseteq U_n \text{ and } \operatorname{diam}_d(U_m) < \epsilon\}$$

□

6.5 The selection theorem

Do for $\mathbb{N}^{\mathbb{N}}$, then use a continuous open surjection to transfer.

6.6 Examples

Countable models of a theory.

Chapter 7

Analytic and coanalytic sets

7.1 Definition and characterizations

Definition 7.1. Let X be a Polish space. A subset $A \subseteq X$ is **analytic** if there is a continuous map $f = \mathbb{N}^{\mathbb{N}} \to X$ such that $A = f(\mathbb{N}^{\mathbb{N}})$. It is **coanalytic** if its complement is analytic.

We denote by $\mathbf{\Sigma}^1_1(X)$ the set of analytic subsets of a Polish space X, and by $\mathbf{\Pi}^1_1(X)$ the set of coanalytic subsets of X.

Example 7.2. By Corollary 6.7, every Borel subset of a Polish space is analytic. Since complements of Borel sets are Borel, every Borel subset of a Polish space is also coanalytic.

One of the goals of this chapter will be to show that the previous examples do not exhaust the class of (co)analytic sets (Thm. **??**). Moreover, we will see that a subset of a Polish space is Borel if and only if it is both analytic and coanalytic (Thm. **??**). But first, we need a better understanding of analytic subsets.

Proposition 7.3. *Let Y be a Polish space and let $B \subseteq Y$. The following are equivalent*

(i) B is analytic;

(ii) there is a Polish space X, a Borel subset $A \subseteq X$ and a Borel map $f : X \to Y$ such that $B = f(A)$;

(iii) there is a closed subset F of $\mathbb{N}^{\mathbb{N}} \times Y$ such that $B = \pi_Y(F)$.

Proof. (iii)⇒(ii) by considering $X = \mathbb{N}^{\mathbb{N}} \times Y$, $A = F$ and $f = \pi_Y$.

For (i)⇒(iii), consider a continuous function $f : \mathbb{N}^{\mathbb{N}} \to Y$ such that $B = f(\mathbb{N}^{\mathbb{N}})$. Then the graph of f is a closed subset of $\mathbb{N}^{\mathbb{N}} \times Y$ whose projection onto Y is equal to $f(\mathbb{N}^{\mathbb{N}}) = B$.

Let us show (ii)⇒(i). Suppose there is a Polish space X, a Borel subset $A \subseteq X$ and a Borel map $f : X \to Y$ such that $B = f(A)$. By Prop. **??** we can change the topology of X without changing its Borel σ-algebra so that f is actually continuous. Then by Cor. **??** we have $g : \mathbb{N}^{\mathbb{N}} \to X$ continuous such that $A = g(\mathbb{N}^{\mathbb{N}})$. Then $f \circ g : \mathbb{N}^{\mathbb{N}} \to Y$ is continuous and $f \circ g(\mathbb{N}^{\mathbb{N}}) = f(A) = B$ so B is analytic. □

Observe that Condition (ii) provides a purely Borel definition of an analytic subset. In particular, being an analytic subset of a Polish space does not really depend on the ambient topology, but rather on the σ-algebra of Borel sets it generates. This motivates the following definition.

Definition 7.4. Let Y be a standard Borel space, then a subset $B \subseteq Y$ is **analytic** if there is a a standard Borel space X, a Borel subset $A \subseteq X$ and a Borel map $f : X \to Y$ such that $B = f(A)$.

By condition (ii) of the previous proposition, a subset of a standard Borel space is analytic if and only if it is analytic for some Polish topology compatible with the Borel σ-algebra, if and only if it is analytic for every Polish topology compatible with the Borel σ-algebra.

Proposition 7.5. *The class of analytic subsets of a Polish space is stable under:*

- *Borel direct image: if $A \in \mathbf{\Sigma}^1_1(X)$ and $f : X \to Y$ is Borel then $f(A) \in \mathbf{\Sigma}^1_1(Y)$;*
- *Borel preimage: if $B \in \mathbf{\Sigma}^1_1(Y)$ and $f : X \to Y$ is Borel $f^{-1}(B) \in \mathbf{\Sigma}^1_1(X)$;*
- *countable intersection: if for every $n \in \mathbb{N}$, $A_n \in \mathbf{\Sigma}^1_1(X)$ then $\bigcap_{n\in\mathbb{N}} A_n \in \mathbf{\Sigma}^1_1(X)$;*
- *countable reunion: if for every $n \in \mathbb{N}$, $A_n \in \mathbf{\Sigma}^1_1(X)$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathbf{\Sigma}^1_1(X)$.*

Proof. The stability under Borel direct image is a straightforward consequence of the characterization of analytic sets as Borel direct images of Borel subsets (item (ii) from the previous proposition).

For stability under countable reunion, let us rather use the original definition of analytic sets. Suppose for each $n \in \mathbb{N}$ we have a continuous map $f_n : \mathbb{N}^\mathbb{N} \to X$ such that $f_n(\mathbb{N}^\mathbb{N}) = A_n$, let us show that $\bigcup_{n\in\mathbb{N}} A_n$ is analytic. Consider the map $f : \mathbb{N} \times \mathbb{N}^\mathbb{N} \to X$ defined by $f(n, x) = f_n(x)$. Then the image of f is equal to $\bigcup_{n\in\mathbb{N}} A_n$, and since the domain of f is $\mathbb{N} \times \mathbb{N}^\mathbb{N}$ which is homeomorphic to $\mathbb{N}^\mathbb{N}$, we conclude $\bigcup_{n\in\mathbb{N}} A_n$ is analytic.

Let us now see why a countable intersection of analytic sets is analytic. We will use a variant of the intersection-to-product trick B. Suppose for each $n \in \mathbb{N}$ we have a continuous map $f_n : \mathbb{N}^\mathbb{N} \to X$ such that $f_n(\mathbb{N}^\mathbb{N}) = A_n$, and consider the map $f : (\mathbb{N}^\mathbb{N})^\mathbb{N} \to X^\mathbb{N}$ which maps $(x_n)_{n\in\mathbb{N}} \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$ to $(f_n(x_n))_{n\in\mathbb{N}}$. Then f is continuous, and its image is $\prod_{n\in\mathbb{N}} A_n$. Now consider the set $F \subseteq (\mathbb{N}^\mathbb{N})^\mathbb{N}$ consisting of all sequences $(x_n)_{n\in\mathbb{N}}$ such that for every $k, l \in \mathbb{N}$ we have $f_k(x_k) = f_l(x_l)$. Observe that F is closed because it is equal to $f^{-1}(\Delta_{X^\mathbb{N}})$ where $\Delta_{X^\mathbb{N}}$ is the closed space of constant sequences in $X^\mathbb{N}$. Then by construction $f(F) = (\bigcap_{n\in\mathbb{N}} A_n)^\mathbb{N}$, so if we let $\pi_1 : X^\mathbb{N} \to X$ be the projection onto the first coordinate, we see that $\bigcap_{n\in\mathbb{N}} A_n = \pi_1 \circ f(F)$. So $\bigcap_{n\in\mathbb{N}} A_n$ is analytic by item (ii) of the previous proposition.

Finally let us prove that the Borel preimage of any analytic set is analytic. Let $f : X \to Y$ be a Borel map, suppose that $A \subseteq Y$ is analytic. Let $\Gamma(f) = \{(x, f(x)) : x \in X\}$ be the graph of f, recall from Lemma 6.8 that it is a Borel subset of $X \times Y$, hence analytic. Now observe that $f^{-1}(A) = \pi_X((X \times A) \cap \Gamma(f))$. The set $X \times A$ is analytic (indeed if $A = f(B)$ with B and f Borel then $X \times A = (\mathrm{id}_X \times f)(X \times B)$) so by the previous part of the proof the intersection $(X \times A) \cap \Gamma(f)$ is analytic. So $f^{-1}(A)$ is the direct image of an analytic set via a Borel map, hence $f^{-1}(A)$ is analytic. □

Exercise 7.1. Donnons maintenant quelques exemples d'ensembles analytiques (les preuves sont laissées en exercice):

- L'ensemble des fonctions continues sur $[0, 1]$ dérivables quelque part, c'est-à-dire des f telles qu'il existe $x_0 \in]0, 1[$ avec f dérivable en x_0.
- L'ensemble des suites de réels qui admettent une sous-suite convergente.

- L'ensemble des compacts de $[0, 1]$ qui contiennent un irrationnel.
- On considère l'ensemble des arbres enracinés localement dénombrables, vu comme fermé de $2^{\mathbb{N}^{<\mathbb{N}}}$. Alors l'ensemble des arbres non bien fondés est analytique.

Par (ii), on a un schéma de Souslin naturel sur A en posant $F_s = f(N_s)$. Comme $f(\mathbb{N}^{\mathbb{N}}) = A$ on a que pour tout n, $\bigcup_{|s|=n} F_s = A$ donc $A = \bigcap_n \bigcup_{|s|=n} F_s$, mais également comme f est continue $A = \bigcup_{y\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} F_{y\restriction n}$

Plus généralement, étant donné un schéma de Souslin (F_s) on définit l'opération de Souslin

$$\mathcal{A}((F_s)) = \bigcup_{y\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} F_{y\restriction n}$$

De manière générale, on a toujours l'inclusion

$$\mathcal{A}((F_s)) \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{|s|=n} F_s$$

Remarquons que l'inclusion peut être stricte: considérons l'homéomorphisme $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{R} \setminus \mathbb{Q}$ et le schéma de Souslin associé (F_s) donné par $F_s = f(N_s)$. Par continuité, c'est un schéma de Souslin évanescent, et donc si on remplace (F_s) par $(\overline{F_s})$, on obtient

$$\mathcal{A}(F_s) = \mathcal{A}(\overline{F_s})$$

mais par densité $\bigcap_{n\in\mathbb{N}} \bigcup_{|s|=n} \overline{F}_s = \mathbb{R}$

Lemma 7.6. *Si le schéma de Souslin est propre, c'est à dire si les F_{si} pour $i \in \mathbb{N}$ sont disjoints pour tout s, alors*

$$\mathcal{A}((F_s)) = \bigcup_{y\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} F_{y\restriction n}$$

Proof. Un élément de l'ensemble droite va, par propreté, définir une unique suite $y \in \mathbb{N}^{\mathbb{N}}$... □

Theorem 7.7. *Soit X polonais non dénombrable. Il existe un (co)analytique non borélien.*

Proof. Soit F un $\mathbb{N}^{\mathbb{N}}$-paramétrage universel des fermés de $\mathbb{N}^{\mathbb{N}} \times X$, c'est à-dire que $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ est fermé et que tout fermé de $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ est de la forme F_x. On définit alors un $\mathbb{N}^{\mathbb{N}}$-paramétrage universel de $\mathbf{\Sigma}^1_1(X)$ en posant $A = \bigcup_{x\in\mathbb{N}^{\mathbb{N}}} \{x\} \times \pi_X(F_x) = \pi_{\mathbb{N}^{\mathbb{N}}\times X}(F)$ qui est donc bien analytique. Alors pour $X = \mathbb{N}^{\mathbb{N}}$, l'antidiagonale de A n'est pas un sous-ensemble analytique de $\mathbb{N}^{\mathbb{N}}$ d'après le théorème de Cantor (**??**). Mais elle est coanalytique par construction, donc on a un coanalytique non borélien, et son complémentaire est analytique non borélien.

Comme $\mathbb{N}^{\mathbb{N}}$ est un G_δ dans tout espace polonais non dénombrable, le théorème est prouvé. □

En regardant le fermé de $\mathbb{N}^{\mathbb{N}^2}$ dont notre analytique non borélien est la projection et en utilisant le fait que $\mathbb{N}^{\mathbb{N}} = \mathbb{R} \setminus \mathbb{Q}$, on trouve un G_δ de $\mathbb{R}^2$ dont la projection n'est pas analytique.

Notons par contre les faits suivants:

- La projection d'un fermé de $\mathbb{R}^2$ est un F_σ. En fait, dans tout polonais K_σ c'est vrai.
- La projection d'un fermé de $K \times X$ où K est compact, est fermée

7.2 The separation theorem

7.3 The analytic graph theorem

Mention that this makes one conclusion in the turning into clopen useless.

7.4 Injective images of Borel sets are Borel

Need some applications apart from the fact that this is a complete characterization of Borel sets.

7.5 Ill-founded trees and complete analytic sets

Every analytic set of $\mathbb{N}^{\mathbb{N}}$ arises as the projection of some closed set F in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. We rather view $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ as $(\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$ so that its closed sets correspond to pruned trees by the previous section. Let T be the unique pruned tree such that $F = [T]$.

We may view T as a set of pairs of sequences (s, t) of the same length, and an end of T as a pair (x, y) of sequences such that for all n we have $(x_{\restriction n}, y_{\restriction n}) \in T$. Now note that $x \in \pi_1([T])$ if and only if F_x is non-empty, which means that we can find y such that for all $n \in \mathbb{N}$ we have $(x_{\restriction n}, y_{\restriction n}) \in T$.

Observe that each $x \in \mathbb{N}^{\mathbb{N}}$ defines a (possibly empty) tree T_x given by $y \in T_x$ if and only if $(x_{\restriction}|y|, y) \in T$. So F_x is non empty if and only if T_x has an end, i.e. if and only if T_x is ill-founded.

Observe that the map $x \mapsto T_x$ is Borel so we have reduced our analytic set to the set of ill-founded trees.

7.6 Well-founded trees and ranks

Soit A un ensemble (qu'on supposera assez souvent dénombrable). Un arbre (enraciné) sur A est un ensemble T non vide de suites finies d'éléments de A qui est stable par préfixe (il contient en particulier la suite vide, qui est sa **racine**). Un arbre T est **bien fondé** s'il n'a pas de branche infinie (appelée aussi bout), c'est-à-dire s'il n'existe pas $x \in A^{\mathbb{N}}$ telle que pour tout n, $x_{\restriction n} \in T$. Par l'axiome du choix dépendant, c'est équivalent à dire que la relation $s \prec t$ si s est un suffixe de t est bien fondée (tout ensemble contient un élément $\prec$-minimal).

Il est taillé s'il n'a pas de feuille, c'est à dire de sommet sans fils. Notons que dans un arbre taillé (sur A dénombrable) tout sommet appartient à une branche infinie, en particulier aucun arbre taillé n'est bien fondé.

7.6.1 WF est $\mathbf{\Pi}^1_1$-complet

Les arbres permettent de mieux comprendre les fermés l'espace de Baire. Il est facile de voir que l'ensemble des bouts d'un arbre, noté $[T]$, est un fermé de $A^{\mathbb{N}}$.

Proposition 7.8. *Tout fermé de l'espace de Baire est l'ensemble des bouts d'un arbre taillé.*

Proof. On considère l'ensemble des s tels que $N_s \cap F \neq \emptyset$. C'est toujours un arbre taillé, mais le fait que F soit fermé nous garantit que $F = [T]$: on a clairement $F \subseteq [T]$ et réciproquement si $x \notin F$ alors on a un ouvert $N_{x\restriction n}$ disjoint de F et donc $x \notin [T]$. □

On considère maintenant les arbres sur $A = B \times C$. On préfère voir un sommet comme un couple de suites finies (b, c) de même longueur. Étant donné $x \in B^{\mathbb{N}}$, on définit $T(x)$ comme l'ensemble des $s \in C^{<\mathbb{N}}$ tels que $(x_{\restriction |s|}, s) \in T$.

On a vu que tout sous-ensemble analytique de $\mathbb{N}^{\mathbb{N}}$ est la projection d'un fermé de $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \simeq (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$. Ce fermé est l'ensemble des bouts d'un arbre (taillé) T sur $(\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$, il s'agit de comprendre sa projection sur la première coordonnée.

Or on a $x \in \pi_1([T])$ ssi il existe y tel que $(x, y) \in [T]$, c'est à dire tel que pour tout n, $(x_{\restriction n}, y_{\restriction n}) \in T$.

Cette condition en y revient à dire que $y \in T(x)$ donc $x \in \pi_1([T])$ ssi $T(x)$ admet une branche infinie, c'est-à-dire si $T(x)$ n'est pas bien fondé.

Ainsi $\pi_1([T])$ se réduit de manière borélienne à IF l'ensemble des arbres sur $\mathbb{N}$ qui ne sont pas bien fondés. Ce dernier est clairement analytique, il est donc $\mathbf{\Sigma}^1_1$-complet (en particulier non borélien puisqu'il existe des analytiques de l'espace de Baire non boréliens). Et donc WF est $\mathbf{\Pi}^1_1$-complet.

7.6.2 Relations bien fondées

Soit X un ensemble et $\prec$ une relation sur X, elle est bien fondée si tout sous-ensemble Y admet un élément minimal y_0, c'est-à-dire que pour tous $y \in Y$, on n'a jamais $y \prec y_0$. En particulier, $\prec$ est antiréflexive.

La propriété fondamentale des ensembles bien fondés est le principe d'induction: pour voir qu'une propriété $P(x)$ est satisfaite par tout élément de X, il suffit de voir que pour tout x, si $P(y)$ est satisfaite pour tout $y \prec x$ alors $P(x)$ est vraie (considérer un élément minimal de l'ensemble des x tels que $P(x)$ est fausse).

Par l'axiome du choix dépendant, une relation est bien fondée ssi il n'y a pas de chaîne infinie descendante (x_n telle que $x_n + 1 \prec x_n$ pour tout n).

7.6.3 Rang d'un arbre bien fondé

Soit T un arbre bien fondé (sur un ensemble A quelconque). La relation naturelle $s \prec t$ ssi t préfixe s et $t \neq s$ est bien fondée, on peut alors définir par induction un rang sur les sommets de T: le rang d'une feuille est zéro, et le rang d'un sommet t est

$$\sup_{s \prec t}\{\rho_T(s) + 1\}$$

Le rang d'un arbre non vide est alors le rang de sa racine +1, et le rang de l'arbre vide est nul. Notons que toutes ces définitions ne font pas intervenir l'axiome du choix.

Plus généralement on peut définir un rang sur une relation bien fondée. Une manière de le faire est d'éplucher un ensemble par l'opération qui retire l'ensemble des minimaux (analogue de la dérivation). On sait qu'il existe un ordinal à partir duquel le procédé stationne: sinon on a une injection de la classe des ordinaux dans l'ensemble des parties de X. De plus une fois que le procédé stationne, on a l'ensemble vide puisque X est bien fondé. Le rang d'un élément est le premier ordinal pour lequel cet élément a été enlevé.

Maintenant, étant donné deux arbres S et T, si il existe $\varphi : S \to T$ strictement croissante et T est bien fondé alors S aussi. Cela provient de la caractérisation en termes de chaînes infinies décroissantes. De plus, on a alors $\rho(S) \leqslant \rho(T)$ car on montrer par une induction immédiate que pour tout s, $\rho_S(s) \leqslant \rho_T(\varphi(s))$.

Pour la réciproque, on va avoir besoin de choix dépendant et de travailler avec S arbre sur $\mathbb{N}$ (on utilise le choix dénombrable pour passer de n à $n+1$ et le choix dépendant nous dit qu'on a quelque chose de bien défini pour tout n). On part de $f(\emptyset) = \emptyset$, puis f étant construite pour $|s| \leqslant n$ avec $\rho_S(s) \leqslant \rho_T(f(s))$, on a pour $a \in \mathbb{N}$ tel que $sa \in S$,

$$\rho_S(sa) < \rho_S(s) \leqslant \rho_T(f(s)) = \sup_b(\rho_T(f(s)b) + 1)$$

En particulier on trouve b tel que $\rho_T(f(s)b) + 1 > \rho_S(sa)$ et donc $\rho_T(f(s)b) \geqslant \rho_S(sa)$, et on envoie sa sur $f(s)b$.

Soit maintenant $\prec$ bien fondée sur X, on lui associe un arbre sur X dont les sommets sont les suites $(x_0, ..., x_n)$ avec $x_n \prec x_{n-1} \cdots \prec x_0$ (et les (x_0) et la suite vide).

Par construction, cet arbre est bien fondé, de plus son rang est celui de la relation. Montrons en effet par induction sur $\prec$ que pour tout x, pour tout n si $x \prec x_n \prec \cdots \prec x_0$ le rang du sommet correspondant dans $T_\prec$ est le rang de x.

Si x est minimal, alors les sommets correspondants sont des feuilles, donc bien de rang 0.

Supposons que ce soit vrai pour les prédécesseurs de x, considérons $x \prec x_n \prec \cdots \prec x_0$. Si x a un prédécesseur y, le rang de x est le sup des rangs de $y \prec x \prec x_n ... \prec x_0 + 1$, qui est le sup des $\rho_\prec(y) + 1$, c'est ce qu'on veut.

Supposons maintenant que X soit un borélien standard. Si le rang de $\prec$ est $\geqslant \omega_1$, on sait que tous les arbres bien fondés sur $\mathbb{N}$ s'envoient via une application strictement croissante dans $T_\prec$.

Autrement dit WF est l'ensemble des S tels qu'il existe une application $f : \mathbb{N}^{<\mathbb{N}} \to X^{<\mathbb{N}}$ (appartenant donc à un espace polonais) telle que pour tout $s, s' \in S$ avec $s \prec s'$, on ait $f(s) \prec f(s')$.

Lemma 7.9. *Le rang d'un arbre sur $\mathbb{N}$ est borné par ω_1.*

Proof. Par induction. On ne prend que des supremum dénombrables. □

Les arbres bien fondés de rang $\leqslant \alpha$ forment un borélien. En effet, on montre d'abord que le rang d'un arbre, c'est le supremum des rangs des sous arbres partant des sommets n pour $n \in \mathbb{N}$.

Alors un arbre est de rang $\leqslant \alpha$ ssi il tous les arbres partant des sommets n sont de rang $\leqslant \beta$ pour des $\beta < \alpha$. Comme l'application $T \mapsto T_{\langle n \rangle}$ est continue, le résultat suit par induction.

Corollary 7.10. *Tout ensemble coanalytique s'écrit comme réunion croissante de ω_1 boréliens.*

On en déduit qu'un ensemble coanalytique est soit de cardinal $\aleph_0$, $\aleph_1$ ou $2^{\aleph_0}$.

7.7 Analytic sets and the Souslin operator

In this section we will define and study basic properties of the Souslin operator, which we have already seen without giving it a proper name. Recall that a Souslin scheme is simply a family $(P_s)_{s \in \mathbb{N}^\mathbb{N}}$ of sets. The **Souslin operator** $\mathcal{A}$ is then defined by

$$\mathcal{A}((P_s)_{s \in \mathbb{N}^{<\mathbb{N}}}) = \bigcup_{y \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} P_{y \restriction n}.$$

The following proposition is a reformulation of a fact that we used several times.

Proposition 7.11. *Let X be a Polish space. Every analytic subset A of X is of the form $A = \mathcal{A}((P_s)_{s\in\mathbb{N}^{<\mathbb{N}}})$ where $(P_s)_{s\in\mathbb{N}^{\mathbb{N}}}$ is a Souslin scheme consisting of closed subsets.*

Proof. Let $f : \mathbb{N}^{\mathbb{N}} \to X$ be a continuous function such that $A = f(\mathbb{N}^{\mathbb{N}})$ and let $P_s := \overline{f(N_s)}$. For every $y \in \mathbb{N}^{\mathbb{N}}$ the point $(f(y))$ has a neighborhood basis consisting of closed sets, and since $(N_{y\restriction n})_{n\in\mathbb{N}}$ is a neighborhood basis of y we have that

The fact that f is continuous implies that for every $y \in \mathbb{N}^{\mathbb{N}}$ □

Chapter 8

Baire measurability

8.1 Nowhere dense sets and meager sets

Recall that the Baire category theorem states that if X is a Polish space and A is a countable reunion of closed subsets of empty interior, then A itself has empty interior. We may thus view countable reunions of closed subsets as small subsets, which motivates the following definition.

Definition 8.1. A subset A of a topological space X is called **meager** if there is a countable family $(F_n)_{n\in\mathbb{N}}$ of closed subsets of empty interior such that $A \subseteq \bigcup_{n\in\mathbb{N}} F_n$. It is called **comeager** if its complement is meager.

Observe that meager subsets form a σ-ideal, meaning that any subset of a meager set is meager and that any countable reunion of meager sets is meager. Moreover, every countable intersection of comeager subsets is comeager, and every set containing a comeager set must be comeager.

Exercise 8.1. Prove the previous observation. Then show that if X is a Polish space, a subset of X cannot be both meager and comeager.

In this section, we will show that the notion of a meager set relativizes well to open subspaces: a subset of an open subspace is meager as a subset of the subspace if and only if it is meager as a subset of the space. Observe however that being closed with empty interior does not relativize well because a closed subset of an open subset may not be closed in the ambient space. The following notion will allow us to overcome this problem.

Definition 8.2. Let X be a topological space. A subset A of X is called **nowhere dense** if for every nonempty open subset U of X, there is a non empty open subset $V \subseteq U$ such that $A \cap V = \emptyset$.

Example 8.3. Let O be an open subset of a topological space X. Then $\overline{O} \setminus O$ is nowhere dense: if U is open, either $U \cap \overline{O} = \emptyset$ and then $V := U$ is already disjoint from $\overline{O} \setminus O$, or $U \cap \overline{O} \neq \emptyset$ in which case we can take $V := O \cap U$ which is nonempty by definition of the closure.

Proposition 8.4. *A subset of a topological space is nowhere dense if and only if its closure has empty interior.*

Proof. Let A be a nowhere dense subset of a topological space X. Observe that if U is an open subset contained in $\overline{A}$, then by definition of the closure any non empty open subset V of U intersects A, so $\overline{A}$ as empty interior.

Conversely, suppose $\overline{A}$ has empty interior. Let U be an open subset of X. If U is disjoint from A there is nothing to do, otherwise U intersects $\overline{A}$ but cannot be contained in $\overline{A}$ because $\overline{A}$ has empty interior. So $V := U \setminus \overline{A}$ is a nonempty open subset of U disjoint from A as wanted. □

Lemma 8.5. *Let X be a topological space, let $A \subseteq Y$. The following hold.*

(1) If A is nowhere dense as a subset of Y then A is nowhere dense as a subset of X.

(2) If A is meager as a subset of Y then A is meager as a subset of X.

Proof. Let us first prove (i). Suppose A is nowhere dense as a subset of Y. Let $U \subseteq X$ be open nonempty. If $U \cap A$ is empty, there is nothing to do. Otherwise, the set $U \cap Y$ is a nonempty open subset of Y, and we thus find $V \subseteq U$ open in Y such that $V \cap A = \emptyset$. By definition of the induced topology there is $W \subseteq X$ open such that $V = Y \cap W$. Then $V' := W \cap U$ is a nonempty open subset of U which is disjoint from A. This proves that A is nowhere dense as a subset of Y.

Now (ii) is a straightforward consequence from (i): suppose A meager in Y and write $A \subseteq \bigcup_{n\in\mathbb{N}} F_n$ where each F_n is nowhere dense in Y, then each F_n is nowhere dense in X so A is also meager in X. □

Remark 8.6. Rather than saying A is nowhere dense/meager as a subset of Y, we will now rather say that A is nowhere dense/meager *in* Y for the sake of simplicity.

Here is an application of the previous lemma. A **Baire topological space** is a topological space in which every meager set has empty interior. The Baire category theorem asserts that every Polish space is a Baire space. The class of Baire topological spaces satisfies the following closure property.

Theorem 8.7. *Let X be a Baire topological space, let $Y \subseteq X$ be open. Then Y is also a Baire topological space for the induced topology.*

Proof. Let $A \subseteq Y$ be meager in Y, then A is also meager in X by the previous lemma. It follows that A has empty interior in X, but since Y is open this implies A has empty interior in Y. □

Let us conclude this section by mentioning that meagerness actually relativises well.

Proposition 8.8. *Let X be a topological space, let $O \subseteq X$ be open, and let $A \subseteq O$. The following hold.*

(1) A is nowhere dense in O if and only if A is nowhere dense in X.

(2) A is meager in O if and only if A is meager in X.

Proof. (1) By Lemma 8.5, if A is nowhere dense as a subset of O then it is nowhere dense as a subset of X. Conversely, suppose A is nowhere dense as a subset of X. If U is an open nonempty subset of O then U is also open as a subset of X, so there is $V \subseteq U$ nonempty open in X such that $V \cap A = \emptyset$. But V is also open in O so we conclude A is nowhere dense as a subset of O.

(2) We already saw that if A is a meager subset of O then A is meager as a subset of X (Lemma 8.5). Conversely, if A is meager as a subset of X then $A \subseteq \bigcup_{n\in\mathbb{N}} F_n$ where each F_n is nowhere dense in X. By item (1) for each n the set $F_n \cap O$ is still nowhere dense, so we conclude $A \subseteq \bigcup_{n\in\mathbb{N}} F_n \cap O$ is meager in O. □

8.2 Baire measurability

Whenever A, B are subsets of a topological space X, we write $A =^* B$ if $A \triangle B$ is meager.

Proposition 8.9. *Let X be a topological space. Then $=^*$ satisfies the following properties:*

(1) $=^*$ is an equivalence relation;

(2) for all $A, B \subseteq X$ we have $A =^* B$ if and only if $X \setminus A =^* X \setminus B$;

(3) if for every $n \in \mathbb{N}$ we have $A_n =^* B_n$ then $\bigcup_{n\in\mathbb{N}} A_n =^* \bigcup_{n\in\mathbb{N}} B_n$.

Proof. Let us first prove (1). The relation $=^*$ is clearly reflexive and symmetric, moreover it is transitive because of the inclusion $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$ and the fact that the reunion of two meager subsets is meager. So $=^*$ is an equivalence relation.

(2) follows directly from the equality $(X \setminus A) \triangle (X \setminus B) = A \triangle B$.

We finally prove (3). Suppose $A_n =^* B_n$ for every $n \in \mathbb{N}$, then by definition $A_n \triangle B_n$ is meager for every $n \in \mathbb{N}$. Observe that

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right) \triangle \left(\bigcup_{n\in\mathbb{N}} B_n\right) \subseteq \bigcup_{n\in\mathbb{N}} A_n \triangle B_n,$$

so since any countable reunion of meager sets is meager, we conclude that $\left(\bigcup_{n\in\mathbb{N}} A_n\right) \triangle \left(\bigcup_{n\in\mathbb{N}} B_n\right)$ is meager, which means that $\bigcup_{n\in\mathbb{N}} A_n =^* \bigcup_{n\in\mathbb{N}} B_n$ as wanted. ☐

Remark 8.10. The above proof only uses the fact that meager subsets form a σ-ideal.

Definition 8.11. Let X be a topological space. A subset A of X is called **Baire-measurable** if there is an open subset $O \subseteq X$ such that $A =^* O$.

Observe that every meager set and every open set is Baire-measurable. Here is a more interesting example.

Example 8.12. We know from Example 8.3 that for every open set O the set $\overline{O} \setminus O$ is nowhere dense, in particular it is meager. We thus have $\overline{O} =^* O$ so the set $\overline{O}$ is Baire-measurable.

Theorem 8.13. *The set $\mathrm{BP}(X)$ of Baire-measurable subsets of a topological space X is a σ-algebra. In particular, it contains all Borel subsets.*

Proof. Since every open set is Baire-measurable, we have $\emptyset \in \mathrm{BP}(X)$.

Let us then show that $\mathrm{BP}(X)$ is stable under complements. Let A be Baire-measurable, then we find $O \subseteq X$ open such that $A =^* O$. We also know that $O =^* \overline{O}$ so by transitivity $A =^* \overline{O}$. Now by taking complements $X \setminus A =^* X \setminus \overline{O}$ and since $X \setminus \overline{O}$ is open we conclude $X \setminus A$ is Baire-measurable.

We then prove that $\mathrm{BP}(X)$ is stable under countable reunions. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of Baire-measurable set. We find a sequence $(O_n)_{n\in\mathbb{N}}$ of open subsets of X such that for every $n \in \mathbb{N}$ we have $A_n =^* O_n$ is meager. By item (3) from the above proposition we gave $\bigcup_{n\in\mathbb{N}} A_n =^* \bigcup_{n\in\mathbb{N}} O_n$. But $\bigcup_{n\in\mathbb{N}} O_n$ is open, so $\bigcup_{n\in\mathbb{N}} A_n$ is Baire-measurable as wanted.

So $\mathrm{BP}(X)$ is a σ-algebra, and since open sets are Baire-measurable, $\mathrm{BP}(X)$ contains all Borel sets. ☐

Proposition 8.14. *Let X be a topological space, let $A \subseteq X$. The following are equivalent*

(1) A is Baire-measurable,

(2) There are a G_δ subset $G \subseteq X$ and a meager subset $M \subseteq X$ such that $A = G \sqcup M$,

(3) There are an F_σ subset $F \subseteq X$ and a meager subset $M \subseteq X$ such that $A = F \setminus M$.

Proof. Both implications (2)⇒(1) et (3)⇒(1) are immediate consequences of the fact that $\mathrm{BP}(X)$ is a σ-algebra which contains Borel and meager sets.

Let us show (1)⇒(2). Let $A \subseteq X$ be Baire-measurable. We have $O \subseteq X$ open such that $A \triangle O$ is meager, which means that $A \triangle O$ is contained in a set $F \subseteq X$ which is a countable reunion of closed subsets of empty interior. Then $G := O \setminus F$ and $M := A \setminus G$ are as wanted (G is disjoint from M by the definition of M, G is G_δ because O is open and F is F_σ, and M is meager because $G =^* O =^* A$).

Finally, let us show that (1)⇒(3). Let $A \subseteq X$ be Baire measurable, then $X \setminus A$ is Baire-measurable. Since the implication (1)⇒(2) holds, we have a G_δ subset $G \subseteq X$ and a meager subset $M \subseteq X$ such that $X \setminus A = G \sqcup M$. Let $F = X \setminus G$, then F is F_σ and we have $A = X \setminus (X \setminus A) = X \setminus (G \setminus M) = F \sqcup M$ as wanted. □

Define Berstein set and show these are not Baire-measurable

8.3 The open enveloppe of a subset

When $A \subseteq U$ is comeager in U, the set A is thought of as a big subset of U. When A is not a subset of U, we can still consider its intersection with U and see wether it is comeager in U or not. Let us give a name to this property.

Definition 8.15. Let X be a topological space, let $U \subseteq X$ be open and let $A \subseteq X$. We say A is **generic** in U if $U \setminus A$ is meager in X.

Proposition 8.16. *Let X be a topological space, let $U \subseteq X$ be open and let $A \subseteq X$. The following are equivalent:*

(1) A is generic in U;

(2) $U \setminus A$ is meager in U;

(3) $U \cap A$ is comeager in U.

Proof. The equivalence between (1) and (2) is a direct consequence of the fact that meagerness relativizes well to open subsets (Proposition 8.8). Moreover (2) is equivalent to (3) by the definitions of meagerness and comeagerness. □

Remark 8.17. The terminology "A is comeager in U" is sometimes also used to say that $A \cap U$ is comeager in U. We prefer to stick to our original definition of comeagerness, which only makes sense for subsets of U.

Definition 8.18. Let X be a topological space, let $A \subseteq X$. Define the **open enveloppe** of A as

$$U(A) := \bigcup \{U \subseteq X \text{ open: } A \text{ generic in } U\}.$$

We will now see that $U(A)$ is the biggest open subset in which A is generic, .

Proposition 8.19. *Let X be a second-countable topological space, let $A \subseteq X$. Then A is generic in $U(A)$.*

Proof. By Lindelöf's lemma, we can find a countable family of open subsets U_n such that A is generic in each U_n and $U(A) = \bigcup_{n\in\mathbb{N}} U_n$. Then for each $n \in \mathbb{N}$ the set $U_n \setminus A$ is meager in X, so $\bigcup_{n\in\mathbb{N}}(U_n \setminus A)$ is meager in X. Since $\bigcup_{n\in\mathbb{N}}(U_n \setminus A) = (\bigcup_{n\in\mathbb{N}} U_n) \setminus A = U(A)$ we conclude that A is generic in $U(A)$. □

Remark 8.20. The above proposition actually also holds in the non second-countable case, using the axiom of choice (see Exercise **??**).

Proposition 8.21. *Let X be a second-countable topological space. The following hold.*

(1) $A =^* U(A)$ if and only if A is Baire-measurable.

(2) If A is non meager and Baire-measurable, then $U(A)$ is nonempty.

Proof. We first prove (1). If $A =^* U(A)$ then A is Baire-measurable because $U(A)$ is open. Conversely, if A is Baire-measurable, let O be an open subset such that $A =^* O$. Then $O \setminus A$ is meager so $O \subseteq U(A)$. We deduce that $A \setminus U(A) \subseteq A \setminus O$ is meager because $A =^* O$. But $U(A) \setminus A$ is meager by the previous proposition so $A \triangle U(A)$ is meager, i.e. $A =^* U(A)$.

Now (2) is a straightforward consequence of (1) because if A is non meager, $U(A) =^* A$ so $U(A)$ cannot be meager. □

Proposition 8.22. *Let X be a second countable Baire topological space. Let A be a Baire-measurable subset of X. Then A is non meager if and only if $U(A)$ is nonempty.*

Proof. The direct implication was established in the previous proposition. We prove the converse by contrapositive. Suppose A is meager, let U be an open subset of X, then $U \setminus A$ is comeager in U. Since X is a Baire topological space, U also is by Theorem 8.7. So A cannot meager in U because otherwise the empty set would be comeager in U. We conclude that A is comeager in no open subset of X, so by definition $U(A) = \emptyset$. □

8.4 Category quantifiers and the Kuratowski-Ulam theorem

Definition 8.23. Let X be a topological space, let $P(x)$ be a property.

We say that $P(x)$ **holds generically**, and we write $\forall^* x\, P(x)$ when the set $\{x \in X : P(x)\}$ is comeager.

We say that $P(x)$ **holds non-meagerly**, and we write $\exists^* x\, P(x)$ when the set $\{x \in X : P(x)\}$ is non-meager.

Let us make a two important remarks which will get us accustomed to this terminology. First, observe that if $P(x)$ and $Q(x)$ are properties such that $\forall^* x\, P(x)$ and $\exists^* x\, Q(x)$, then there is $x \in X$ such that $P(x)$ and $Q(x)$ hold. Indeed, the set $A := \{x \in X : P(x)\}$ is comeage while $B := \{x \in X, Q(x)\}$ is non meager so they must intersect (if not, then B is contained in the complement of A, hence B is meager !).

Second, we have the following *universal quantifier exchange rule*: if $(P_n(x))_{n\in\mathbb{N}}$ is a sequence of properties, then

$$\forall n \forall^* x\, P_n(x) \iff \forall^* x \forall n\, P_n(x)$$

Indeed if $A_n := \{x \in X : P_n(x)\}$, the above equivalence says that A_n is comeager for every $n \in \mathbb{N}$ if and only if $\bigcap_{n\in\mathbb{N}} A_n$ is comeager, which is true because countable intersections of comeager sets are comeager.

The Kuratowski-Ulam theorem can be thought of as the topological version of Fubini's theorem, which in its most basic form says that a measurable subset of a product space has full measure for the product measure if and only if almost all its fibers have full measure. Here, we start by proving a similar statement for open dense subsets.

Lemma 8.24. *Let X be a topological space, let Y be a second-countable topological space. Suppose $U \subseteq X \times Y$ is open dense. Then $\forall^* x$ the set U_x is open dense.*

Proof. First observe that since U is open, U_x is open for *every* $x \in X$. So we only need to show that $\forall^* X\, U_x$ is dense. Let $(V_n)_{n\in\mathbb{N}}$ be a countable basis of the topology of Y consisting of nonempty open sets, we then need to show $\forall^* x \forall n\, U_x \cap V_n \neq \emptyset$. By the universal quantifier exchange rule, it suffices to show that for every $n \in \mathbb{N}$, the set $\{x \in X : U_x \cap V_n \neq \emptyset\}$ is comeager.

So fix $n \in \mathbb{N}$ and consider the open set $X \times V_n$. Because U is dense in $X \times Y$ and $X \times V_n$ is open, the open set $U \cap (X \times V_n)$ is dense in $X \times V_n$. So its projection onto X is open dense as well, which means $\{x \in X : U_x \cap V_n \neq \emptyset\}$ is open dense. In particular, $\{x \in X : U_x \cap V_n \neq \emptyset\}$ is comeager as wanted. □

We now upgrade the above lemma to obtain the full analogue of the fact that a measure zero set in a product space has almost all its fibers of measure zero.

Lemma 8.25. *Let X be a topological space, let Y be a second-countable topological space. The following hold.*

(1) If $A \subseteq X \times Y$ is meager, then $\forall^* x$ the vertical section A_x is meager in X.

(2) If $A \subseteq X \times Y$ is comeager, then $\forall^* x$ the vertical section A_x is comeager in X.

Proof. First observe that (1) and (2) are equivalent because one can be obtained from the other by taking complements.

We thus only need to prove that (2) holds. Suppose $A \subseteq X \times Y$ is comeager. We find a countable family (U_n) of open dense subsets of $X \times Y$. For every $n \in \mathbb{N}$, the previous lemma yields that $\forall^* x$ the vertical section $(U_n)_x$ is comeager. Applying the universal quantifier exchange rule, we deduce that $\forall^* x$ the vertical section $(U_n)_x$ is comeager for every $n \in \mathbb{N}$. Since for every x the section A_x contains $\bigcap_{n\in\mathbb{N}} (U_n)_x$ and countable intersections of comeager sets are comeager, we conclude that $\forall^* x$ the section A_x is comeager as wanted. □

The definition of the product measure implies that a product of measurable sets $A \times B$ has measure zero if and only if one of the factors has measure zero. We now prove the analogous fact in the topological context.

Lemma 8.26. *Let X and Y be second-countable topological spaces, let $A \in \mathrm{BP}(X)$ and $B \in \mathrm{BP}(Y)$. Then $A \times B$ is meager if and only if A is meager or B is meager.*

Proof. If A is meager, write $A \subseteq F_n$ where each F_n is closed with empty interior. Then for each $n \in \mathbb{N}$ the set $F_n \times Y$ is closed in $X \times Y$ and has empty interior (see Exercise **??**). The set $A \times Y = \bigcup_{n\in\mathbb{N}} F_n \times Y$ is thus meager, in particular $A \times B$ is meager. The same argument shows that if B is meager then $A \times B$ is meager, which finishes the proof of the reverse implication.

Now suppose $A \times B$ is meager but A is non meager. Then by the above lemma we have $\forall^* x\, (A \times B)_x$ is meager, and since A is non meager we find $x \in A$ such that $(A \times B)_x$ is meager, which means B is meager as wanted. □

Before we start proving the Kuratowski-Ulam theorem, we mention one useful consequence of the direct implication from the previous lemma.

Proposition 8.27. *Let X and Y be second-countable topological spaces. If $A \in \mathrm{BP}(X)$ and $B \in \mathrm{BP}(Y)$ then $A \times B \in \mathrm{BP}(X \times Y)$*

Proof. Let $U \subseteq X$ open such that $A =^* U$ and $V \subseteq Y$ open such that $B =^* V$. We have the following inclusion:

$$(A \times B) \triangle (U \times V) \subseteq (A \triangle U) \times Y \cup X \times (B \triangle V)$$

But by the previous lemma, both $(A \triangle U) \times Y$ and $X \times (B \triangle V)$ are meager so $(A \times B) \triangle (U \times V)$ is meager. Since $U \times V$ is open, we conclude $A \times B$ is Baire-measurable. □

Theorem 8.28. *Let X and Y be second-countable topological spaces. Let $A \in \mathrm{BP}(X \times Y)$. The following hold.*

(1) A is comeager $\iff \forall^* x\ A_x$ is comeager $\iff \forall^* y\ A^y$ is comeager.

(2) A is meager $\iff \forall^* x\ A_x$ is meager $\iff \forall^* y\ A^y$ is meager.

(3) $\forall^* x\ A_x \in \mathrm{BP}(Y)$ and $\forall^* y\ A^y \in \mathrm{BP}(X)$.

8.5 BP(X) has enveloppes

In this section we will that BP(X) has enveloppes and thus is stable under the Souslin operator.

8.6 Meager relations and the Kuratowski-Mycielski theorem

8.7 Applications to equivalence relations

A meager equivalence relation has all its classes meager. To show an F_σ equivalence relation is meager, need to find two distinct dense equivalence classes. Applications to spaces of probability measures (measures orthogonal to one given measure are dense G_δ: take two disjoint countable dense sets and look at measures "supported" on these sets). Need to explain somewhere why proba measures are limits of convex combinations of Dirac (clear via density of step functions in L^∞).

Chapter 9

Polish groups

9.1 Definition and first examples

A **topological group** is a group G equipped with a separated topology τ such that the product map $(g,h) \mapsto gh$ and the inverse map $g \mapsto g^{-1}$ are continuous. The most basic non-discrete example of a topological group is probably the reals equipped with the addition: using the triangle inequality one shows that addition is continuous, and the inverse map $x \mapsto -x$ is actually an isometry, hence continuous.

Another example is given by the group of invertible matrices for the topology induced by $\mathbb{R}^{n^2}$. Indeed, the fact that the formulas for the product and the inverse are rational fraction in terms of the coefficients implies that they are continuous. Note that every subgroup of a topological group is a topological group for the induced topology.

Definition 9.1. A **Polish group** is a topological group whose topology is Polish.

We will now give many examples of such groups, each of which is important in its own right.

9.1.1 Separable Banach spaces

9.1.2 The group $\mathfrak{S}_\infty$ of permutations of the integers

9.2 Non-archimedean Polish groups

9.3 Left-invariant (pseudo)-metrics

When we are given a normed vector space, we can naturally endow it with the metric defined by $d(x,y) = \|x-y\|$. Observe that translations are then isometries for this metric.

The aim of this section is to carry out a similar procedure for Polish groups. We will build a natural analogue of a norm for a group and we will then see two natural ways to extend this to a metric on the whole group. The first will turn left translations into isometries, and the second will turn right translations into isometries.

Definition 9.2. A (pseudo-)metric d on a group G is called **left-invariant** if for all $g,g',h \in G$ we have $d(hg,hg') = d(g,g')$, and **right-invariant** if for all $g,g',h \in G$ we have $d(gh,g'h) = d(g,g')$,

Exercise 9.1. Let G be a non-archimedean Polish group and let (G_n) be a decreasing basis of neighborhoods of the identity consisting of open subgroups. Show that the function $l : G \to \mathbb{R}^+$ defined by

$$l(g) = 2^{-\min\{n\in\mathbb{N}:g\in G_n\}}$$

is a continuous length function on G.

As we said, to obtain a metric from a norm, we let $d(x, y) = \|x - y\|$. In a non-commutative group endowed with a length function l, a natural analogue is to define

$$d(x, y) = l(xy^{-1}).$$

Observe that the function d we obtained satisfies

$$d(x_1y, x_2y) = l(x_1yy^{-1}x_2) = l(x_1x_2^{-1}) = d(x_1, x_2).$$

In other words, for each $y \in G$ the right translation by y (i.e. the map $g \mapsto gy$) is an isometry $(G, d) \to (G, d)$.

We have seen that $\mathfrak{S}_\infty$ is endowed with a compatible metric d defined by

$$d(\sigma, \tau) = 2^{-\min\{n\in\mathbb{N}:\sigma(x)\neq\sigma'(x)\}}.$$

Such a metric is not complete, but let us remark that for all $\sigma, \sigma', \tau \in \mathfrak{S}_\infty$ and $n \in \mathbb{N}$ we have $\tau\sigma(n) \neq \tau\sigma'(n)$ if and only if $\sigma(n) \neq \sigma'(n)$ so that

$$d(\tau\sigma, \tau\sigma') = d(\sigma, \sigma').$$

This means that the metric d is *left-invariant*.

In this section, we will actually endow any Polish group with such a left-invariant metric. For a non-archimedean Polish group G, we can simply use the metric induced by d on G viewed as a closed subgroup of $\mathfrak{S}_\infty$ (cf. Theorem **??**). As a justification for the general construction Let us remark that such a metric could be directly defined by letting (G_n) be a basis of neighborhoods of the identity consisting of open subgroups and then letting

$$d($$

The result behind this is somehow technical. We will first state it and derive some consequences so as to motivate it. The proof will only be given at the end of the section.

Definition 9.3. Let G be a group. Given a subset $A \subseteq G$, we let $A^{-1} = \{g^{-1} : g \in A\}$ and say A is **symmetric** if $A = A^{-1}$.

Given two subsets $A, B \subseteq G$, we denote by $A \cdot B$ the set of products of the form ab for $a \in A$ and $b \in B$. We will also denote by $A^{\cdot n}$ the set of products of elements of A, namely

$$A^{\cdot n} = \{a_1a_2 \cdots a_n : a_1, ..., a_n \in A\}.$$

Remark 9.4. It is pretty common in the litterature to skip the dot and simply write A^n, which has the defect of being ambiguous.

Theorem 9.5. *Let G be a group, let (U_n) be a decreasing family of symmetric subsets of G containing 1_G such that for all $n \in \mathbb{N}$,*

$$U_{n+1}^{\cdot 2} \subseteq U_n.$$

Then there is a pseudo-metric d on G such that *(i) The pseudo-metric d is left invariant.*

(ii) For all $x \in U_n$, $d(1,x) \leqslant 3 \cdot 2^{-n}$

(iii) For all $x \notin U_n$, $d(1,x) \geqslant 2^{-n}$

Proof. First note that by using our hypothesis that for all $n \in \mathbb{N}$, $U_{n+1}^{\cdot 2} \subseteq U_n$, a straightforward induction on k yields that for every $n_0 < n_1 < \cdots < n_k$ we have

$$U_{n_0} \supseteq U_{n_1} U_{n_2} \cdots U_{n_k}. \tag{*}$$

We will now build a natural analogue of a norm in our group G using a construction reminiscent of the Urysohn construction of a continuous test function on a normal topological space (Lem. 1.140). For each $n \in \mathbb{N}^*$ Let $V_{2^{-n}} := U_n$. We will extend this definition so that V_r makes sense for every r positive dyadic fraction. First, for all $r \geqslant 1$ we let $V_r = G$. Next, for $r \in \mathbb{N}^*[1/2]$ and $r < 1$ write the dyadic expansion of r as $r = 2^{-n_1} + \cdots + 2^{-n_k}$ with $n_1 < \cdots < n_k$. Then let

$$V_r = V_{2^{-n_1}} \cdots V_{2^{-n_k}}.$$

Given $r, s \in \mathbb{N}^*[1/2]$, note that if the biggest exponent n_k in the dyadic expansion of r satisfies $2^{-n_k} > s$ then

$$V_{r+s} = V_r V_s$$

Now property (*) may be restated as: for every $n_0 < n_1 < \cdots < n_k$, if we let $t = 2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_k}$ then

$$V_{2^{-n_0}} \supseteq V_t.$$

We will use this property to show that the V_r's are nested, i.e.

$$\forall r < s, V_r \subseteq V_s \tag{9.1}$$

So suppose $r < s$, write $r = 2^{-n_1} + \cdots + 2^{-n_k}$ with $n_1 < \cdots < n_k$ and $s = 2^{-m_1} + \cdots + 2^{-m_l}$ with $m_1 < \cdots < m_l$. Let i be the first integer such that $m_i \neq n_i$, then since $r < s$ we must have $m_i < n_i$. Since $m_i < n_i < n_{i+1} < \cdots n_k$, property (*) yields

$$V_{2^{-m_i}} \supseteq V_{2^{-n_i}} \cdots V_{2^{-n_k}}.$$

Multiplying each side by $V_{2^{-n_1}} \cdots V_{2^{-n_{i-1}}}$ on the left, we finally obtain

$$V_s \supseteq V_{2^{-n_1}} \cdots V_{2^{-n_{i-1}}} V_{2^{-m_i}} \supseteq V_r$$

We now define a function which will behave almost like a norm on G: for $g \in G$ we let

$$\phi(g) := \inf\{r : g \in V_r\}.$$

Then clearly $\phi(g) \leqslant 2^{-n}$ for all $x \in U_n$, and $\phi(g) \geqslant 2^{-n}$ for all $g \notin U_n$. Moreover since each V_r contains 1_G, we have $\phi(1_G) = 0$.

We will now see that ϕ satisfies a uniform continuity-like inequality, which will allow us to build a left-invariant pseudo metric with nice properties out of it.

The key to this inequality an estimate on products of the V_r's, namely for all $r \in \mathbb{N}[1/2]$ and $n \in \mathbb{N}$,

$$V_r V_{2^{-n}} \subseteq V_{r+3 \cdot 2^{-n}} \tag{9.2}$$

First note that if n is stricly larger than the biggest exponent in the dyadic expansion of r, then we have

$$V_rV_{2^{-n}} = V_{r+2^{-n}} \subseteq V_{r+3\cdot 2^{-n}}.$$

If not, write $r = 2^{-n_1} + \cdots + 2^{-n_k}$ with $n_1 < \cdots < n_k$. Let $i \in \mathbb{N}$ such that $n_i < n \leqslant n_{i+1}$ (if $n \leqslant n_1$ we take $i = 0$). Then consider $r' = 2^{-n_1} + \cdots + 2^{-n_i} + 2 \cdot 2^{-n}$. Observe that $r' > r$, $r' - r \leqslant 2 \cdot 2^{-n}$ and n is stricly larger than the biggest exponent in the dyadic expansion of r'. We thus have

$$V_rV_{2-n} \subseteq V_{r'}V_{2^{-n}} = V_{r'+2^{-n}}.$$

Since $r \leqslant 2 \cdot 2^{-n} + r$ we conclude $V_{r'+2^{-n}} \subseteq V_{r+3\cdot 2^{-n}}$ as wanted.

We can now use 9.2 to establish the desired uniform continuity-like inequality, namely

$$\text{for all } x \in V_{2^{-n}} \text{ and all } g \in G, \quad |\phi(gx) - \phi(g)| \leqslant 3 \cdot 2^{-n} \tag{9.3}$$

Equation 9.2 implies that for all $g \in G$ and all $x \in V_{2^{-n}}$, if $\phi(g) < r$, then $\phi(gx) \leqslant r + 3 \cdot 2^{-n}$. Taking the infimum over all r such that $\phi(g) < r$, we conclude that

$$\phi(gx) \leqslant \phi(g) + 3 \cdot 2^{-n}$$

for all $g \in G$ and all $x \in V_{2^{-n}}$. Since $V_{2^{-n}} = U_n$ is symetric, we deduce that for all $g \in G$ and all $x \in V_{2^{-n}}$, we have $\phi(gx^{-1}) \leqslant \phi(g) + 2^{-n}$. Replacing g by gx in the above equation ($g \mapsto gx$ is a bijection of G!), we conclude that for all $g \in G$ and $x \in V_{2^{-n}}$,

$$\phi(g) \leqslant \phi(gx) + 3 \cdot 2^{-n},$$

which finishes the proof of equation 9.3.

We have a nice pseudometric for points close to the identity given by $(x, y) \mapsto |\phi(x) - \phi(y)|$. We will now propagate it to the whole group by letting

$$d(x, y) = \sup_{g \in G} |\phi(gx) - \phi(gy)|.$$

Let us now check our function d is a pseudo-metric. Clearly $d(x, x) = 0$ for all $x \in G$ and $d(x, y) = d(y, x)$ for all $x, y \in G$. For the triangle inequality, note that for all $g, x, y, z \in G$

$$|\phi(gx) - \phi(gz)| \leqslant |\phi(gx) - \phi(gy)| + |\phi(gy) - \phi(gz)| \leqslant d(x, y) + d(y, z).$$

Taking the supremum over $g \in G$ establishes $d(x, z) \leqslant d(x, y) + d(y, z)$.

Finally, let us check that d has the desired properties.

(i) Given $x, y, h \in G$ we have

$$d(hx, hy) = \sup_{g \in G} d(ghx, ghx).$$

Since $g \mapsto g$ is a bijection of G we can replace gh by g in the right-hand term, which establishes left-invariance.

(ii) Given $x \in U_n = V_{2-n}$ and $h \in G$, equation 9.3 implies $|\phi(gx) - \phi(g)| \leqslant 3 \cdot 2^{-n}$ for all $g \in G$, so by taking the supremum $d(1_G, x) \leqslant 3 \cdot 2^{-n}$.

(iii) Given $x \notin U_n = V_{2^{-n}}$ we have $\phi(x) \geqslant 2^{-n}$ and since $\phi(1_G) = 0$ we deduce $|\phi(x) - \phi(1_G)| \geqslant 2^{-n}$. We conclude that $d(1_G, x) \geqslant 2^{-n}$. □

9.4 Building new Polish groups out of old ones

9.5 Some important classes of Polish groups

9.5.1 Locally compact second-countable groups

As we already pointed out, the topological groups $\mathbb{R}$, $\mathbb{R}^n$ or $Gl_n(\mathbb{R})$ are examples of Polish groups. They actually belong to the well-studied class of **locally compact groups**, i.e. topological groups whose topology is locally compact. By Theorem **??**, a locally compact group is Polish if and only if it is second-countable. In particular $\mathbb{R}$, $\mathbb{R}^n$, $Gl_n(\mathbb{R})$ (and more generally Lie groups) are locally compact second-countable groups. More examples provided in the next exercise.

Exercise 9.2.

One of the main features of locally compact groups is the existence of a Haar measure, which is a powerful generalization of the Lebesgue measure on the reals. This actually characterizes locally compact groups among Polish groups by a result of []. For a nice treatment of the Haar measure we refer the reader to [].

Another nice feature of locally compact groups is that the totally disconnected ones are actually non-archimedean, which is a much stronger notion of non-connectedness. The situation is far more complicated (hence interesting !) for Polish groups, cf. Exercise **??**..

Most of the Polish groups that we are going to deal with are very far from being locally compact since they will often have all their compact subsets of empty interior, like the Baire space $\mathbb{N}^{\mathbb{N}}$.

9.6 Continuous actions on Polish spaces

Vaught transform: good motivation end of chapter 1 in Hjorth, Group Actions and Countable Models