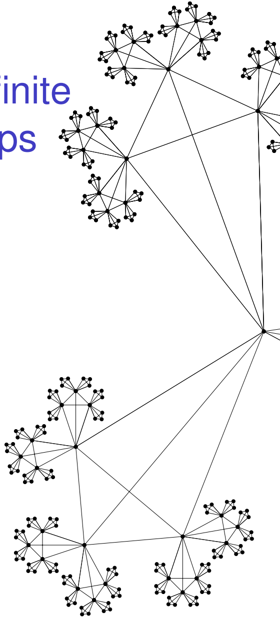
The structure of subdegree finite primitive permutation groups

Simon M. Smith

Polish Groups and Geometry

Paris, June 2016



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When studying infinite permutation groups, one typically wishes to impose some kind of finiteness condition on G

E.g:

- G has only finitely many orbits on Ω^n, for all $n \in \mathbb{N}$ (Oligomorphic)
- G_α has only finite orbits, for all $\alpha \in \Omega$ (Subdegree finite)

Subdegree finite permutation groups are the natural permutation representations of tdlc groups

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- Suppose H is tdlc. By van Dantzig's theorem, H contains a compact open subgroup U
- Let Ω be the set of cosets of U in H, then H acts transitively on Ω by multiplication
- Think of $\mathrm{Sym}(\Omega)$ as a topological group, where the basis of the topology is all pointwise stabilizers of finite subsets of Ω
- Let $H /\!\!/ U$ denote the closure of the permutation group induced by H acting on Ω. Then $H /\!\!/ U$ is subdegree finite
- $H /\!\!/ U$ is called the Schlichting completion of the pair (H, U) by Reid and Wesolek

The wreath product in its product action

Suppose $H \leq \mathrm{Sym}(\Gamma)$ and $m \in \mathbb{N}$

$H \,\mathrm{Wr}\, S_m$ has a product action on Γ^m:

- Think of elements of $H \,\mathrm{Wr}\, S_m$ as $(h_1, \ldots, h_m)\sigma$, where each $h_i \in H$ and $\sigma \in S_m$
- For $(\gamma_1, \ldots, \gamma_m) \in \Gamma^m$ we have

$$(\gamma_1, \ldots, \gamma_m)^{(h_1, \ldots, h_m)\sigma} = (\gamma_1^{h_1}, \ldots, \gamma_m^{h_m})^\sigma$$

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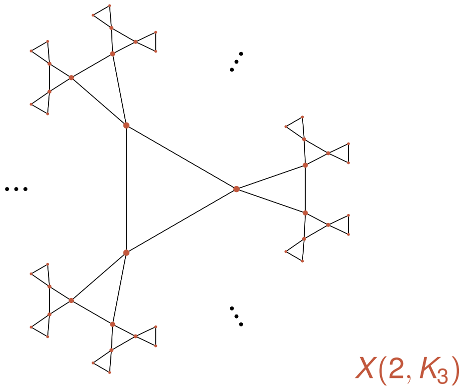
$$= (\gamma_{\sigma^{-1}(1)}^{h_{\sigma^{-1}(1)}}, \ldots, \gamma_{\sigma^{-1}(m)}^{h_{\sigma^{-1}(m)}})$$

The box product

Suppose $H \leq \mathrm{Sym}(\Gamma)$ is transitive and $m \in \mathbb{N}$

Let Λ be a graph whose vertex set is Γ, such that $H \leq \mathrm{Aut}(\Lambda)$

Let $X(m, \Lambda)$ be the (infinite) graph such that every vertex x lies in m copies of Λ, and these copies only intersect at x



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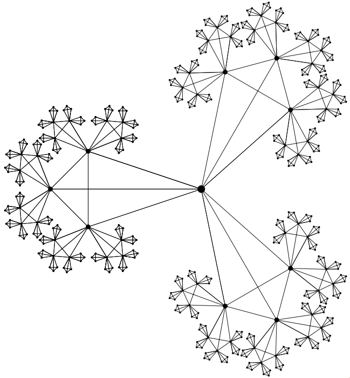

$X(3, K_4)$

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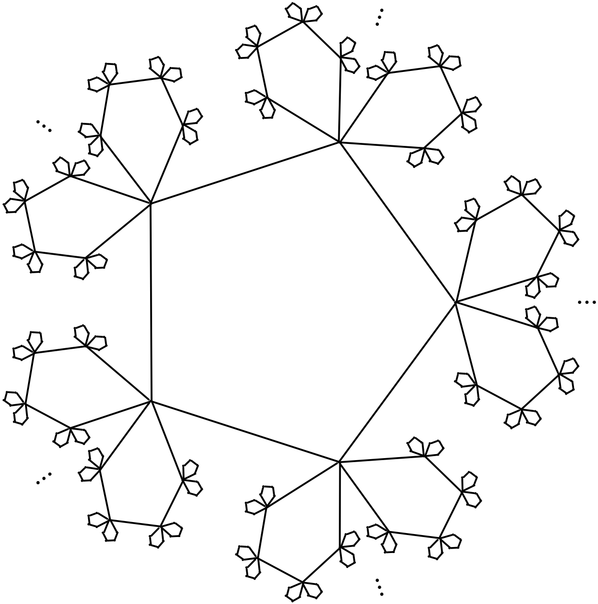
The box product $H \boxtimes S_m$ is the largest transitive subgroup of $\mathrm{Aut}(X(m, \Lambda))$ that induces H on each of the lobes

An intuitive description of $D_{10} \boxtimes S_3$



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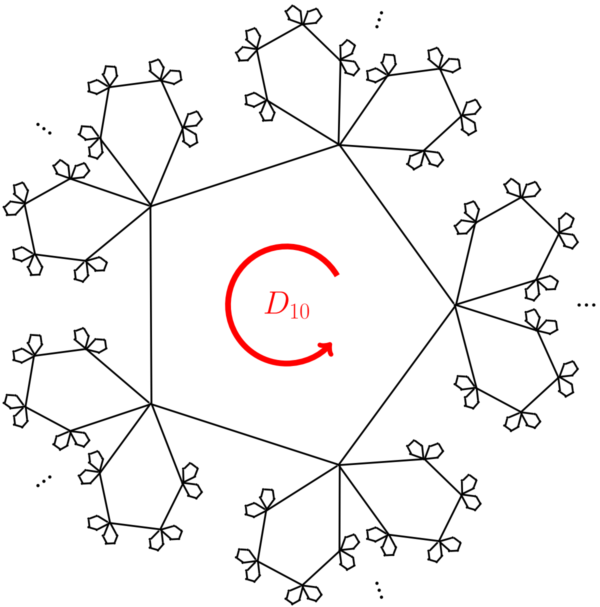

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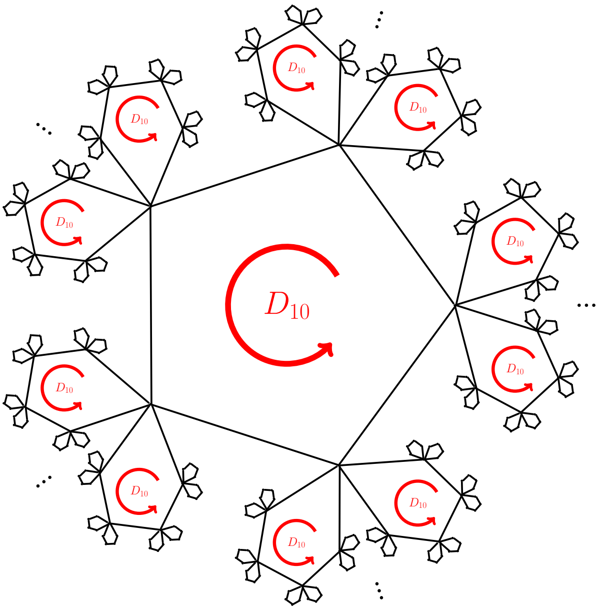

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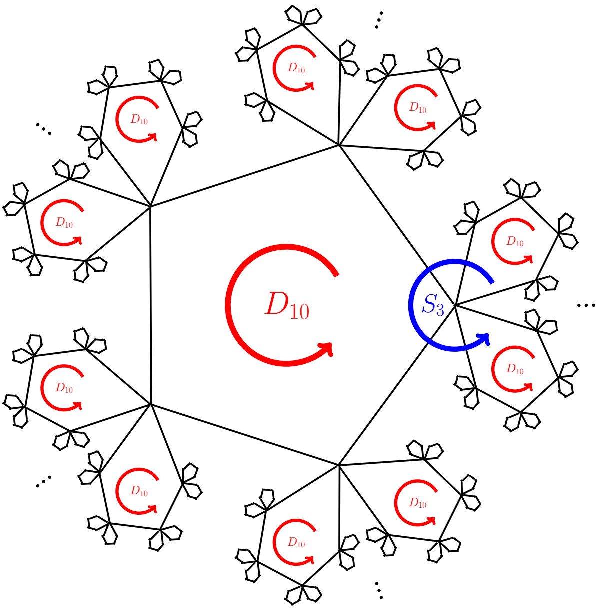

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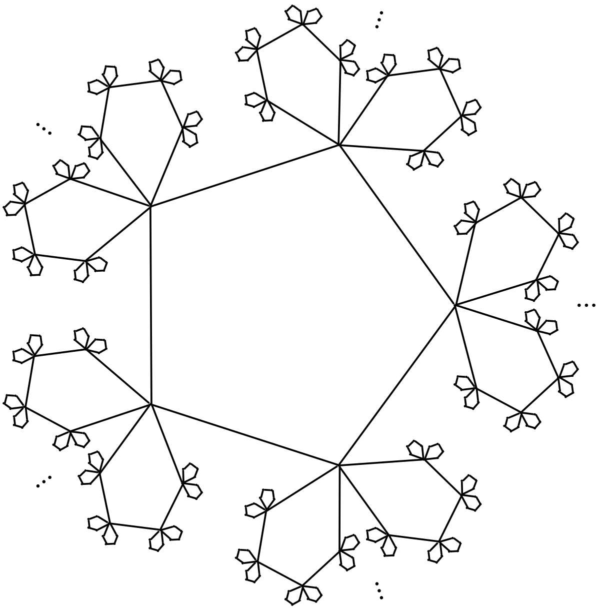

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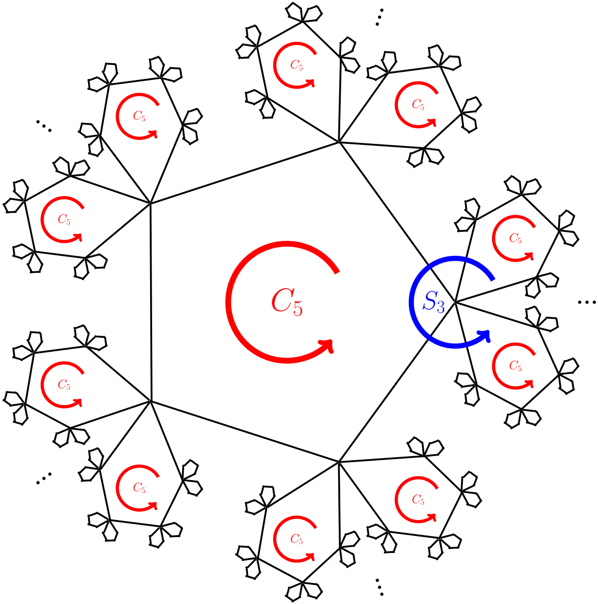

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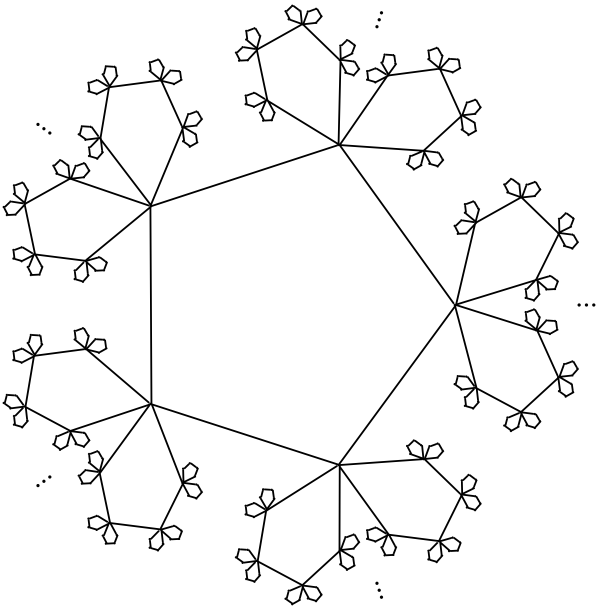

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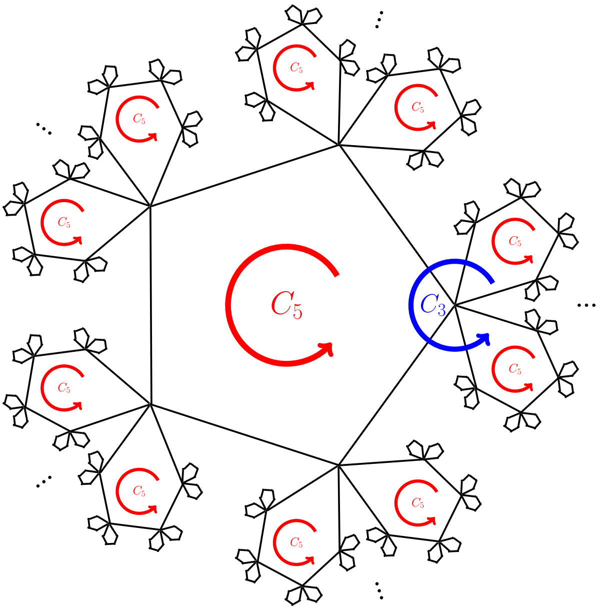

Why is the box product important?

For permutation group theorists it is important because it is the first product that preserves primitivity to have been discovered in over a century

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For topological group theory, the box product was used to prove:

Theorem. (S., 2014) There are $2^{\aleph_0}$ pairwise non-isomorphic, tdlc, compactly generated simple groups.
Moreover, these groups can be chosen so that they contain the same compact open subgroup.

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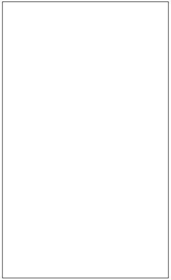

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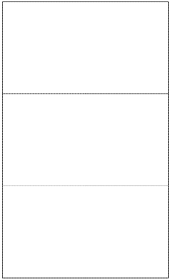

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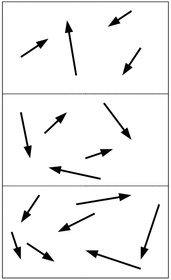

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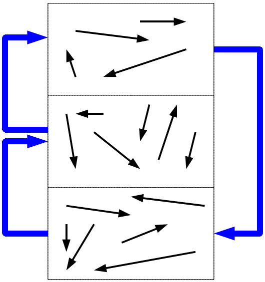

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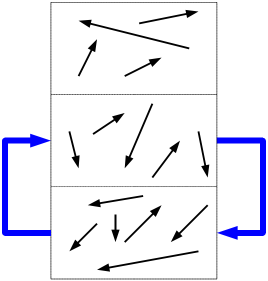

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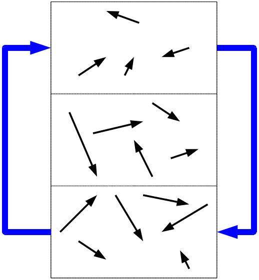

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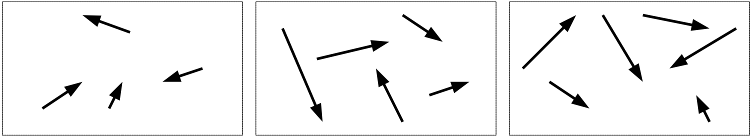

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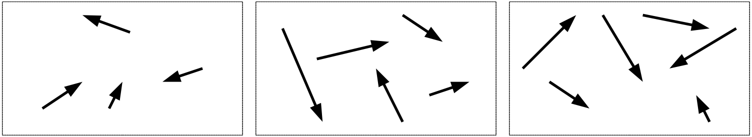

All finite permutation groups can be decomposed into primitive pieces

Finite primitive permutation groups

The structure of finite primitive permutation groups is known

O'Nan-Scott Theorem ('79). Every finite primitive permutation group G is either:

- Basic (affine, almost simple or diagonal)
- Contained in $H \operatorname{Wr} Sym(m)$ with its product action, where H is basic
- (or twisted wreath type)

Structure of subdegree finite primitive permutation groups

Theorem (S.) Suppose $G \leq \mathrm{Sym}(\Omega)$ is closed, infinite, subdegree finite and primitive, then G is:

- Almost topologically simple:
  - (a) Almost simple and discrete
  - (b) Almost topologically simple and non-discrete
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- Given $\alpha, \beta \in \Omega$ distinct, the orbital graph Γ with vertex set Ω and directed edge set $(\alpha, \beta)^G$ is connected
  - If we forget the edge-direction then Γ is a Cayley–Abels graph for G
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- Theorem (S., '10) A primitive subdegree finite permutation group with more than one end is not discrete

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- Hence $G \leq \mathrm{Aut}\,(X(m, \Lambda))$
- Let H be the subgroup of $\mathrm{Aut}\,(\Lambda)$ induced by the setwise stabilizer $G_{\{\Lambda\}}$
- Then H is primitive but not regular and
- G is contained in $H \boxtimes S_m$

Classification of subdegree finite primitive permutation groups

Theorem (S.) Suppose $G \leq \mathrm{Sym}(\Omega)$ is closed, infinite, subdegree finite and primitive, then G satisfies:

- G is almost topologically simple with one end
  - (a) Almost simple and discrete
  - (b) Almost topologically simple and non-discrete
- G has $2^{\aleph_0}$ ends and is contained in $H \boxtimes S_m$, where $m \geq 2$ and H is subdegree finite, primitive (possibly finite) but not regular
- G is contained in $H \,\mathrm{Wr}\, Sym(m)$ with its product action, where H is almost simple (or of box product type), subdegree finite and primitive

Open questions and future work

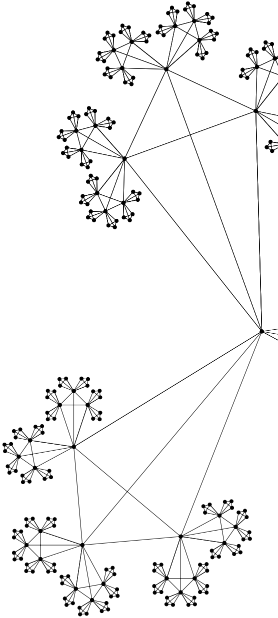
Question 1a: Does there exist a simple, subdegree finite, primitive permutation group that is non-discrete and has precisely one end?

Question 1b: Does there exist a simple non-discrete tdlc second countable group H which contains a compact open (proper) subgroup U such that U is maximal in H and the Cayley–Abels graph of (H, U) has precisely one end?

Question 2: In the box product case we write $G \leq H \boxtimes S_m$. But $H \boxtimes S_m$ is huge; how "small" can G be?

preprint coming soon

thank you



Theorem (poss. attributable to W. Manning, early 20th C)

$M \operatorname{Wr} N$ acting on X^Y with its product action is primitive $\iff$

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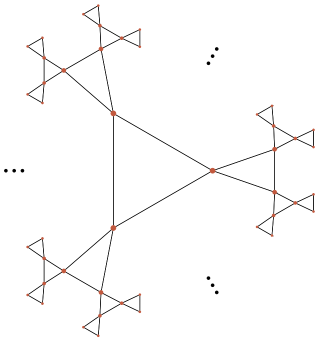
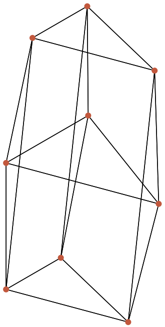
Geometry

One can see the “shape” of a permutation group $G \in \mathrm{Sym}(\Omega)$ by looking at an orbital graph Γ.



$\mathrm{Sym}(3) \boxtimes \mathrm{Sym}(2)$



$\mathrm{Sym}(3)\,\mathrm{Wr}\,\mathrm{Sym}(2)$

Topological properties

Suppose:

$M \leq \mathrm{Sym}(X)$ $N \leq \mathrm{Sym}(Y)$ $M \boxtimes N \leq \mathrm{Sym}(V_Y)$

are given their permutation topologies, & M, N are closed.

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- every point stabiliser in M is compact and N is compact
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Theorem (S.) Suppose M and N are transitive. If

- M is compactly generated and every point stabiliser is compact and
- N is compact

then $M \boxtimes N$ is compactly generated and every point stabiliser is compact

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Theorem (S.) $M \boxtimes N$ is discrete $\iff$ M & N are semi-regular.