The topological conjugacy relation for Toeplitz subshifts

Marcin Sabok

Paris, June 8, 2016

Definition

We say that an equivalence relation E on a Polish space X is *Borel* if E is a Borel subset of $X \times X$.

Definition

We say that an equivalence relation E on a Polish space X is *Borel* if E is a Borel subset of $X \times X$.

Sometimes, Borel equivalence relations arise from Borel actions of countable groups $\Gamma \curvearrowright X$.

Definition

A Borel equivalence relation on a standard Borel space is *countable* if it has countable classes.

Definition

A Borel equivalence relation on a standard Borel space is *countable* if it has countable classes.

By a classical theorem of Feldman–Moore countable Borel equivalence relations are exactly those which arise as Borel actions of countable discrete groups.

Definition

A Borel equivalence relation on a standard Borel space is *countable* if it has countable classes.

By a classical theorem of Feldman–Moore countable Borel equivalence relations are exactly those which arise as Borel actions of countable discrete groups.

In general, the group, although countable, may, however be quite complicated.

Definition

A countable equivalence relation is called *hyperfinite* if it induced by a Borel action of $\mathbb{Z}$.

Definition

A countable equivalence relation is called *hyperfinite* if it induced by a Borel action of $\mathbb{Z}$.

Theorem (Slaman–Steel, Weiss)

For a Borel countable equivalence relation E, the following are equivalent:

- E is hyperfinite,
- E is an increasing union of Borel equivalence relations E_n such that each E_n has finite classes.

Given an equivalence relation E on X and a function $f : E \to \mathbb{R}$, for $x \in X$ denote by $f_x : [x]_E \to \mathbb{R}$ the function $f_x(y) = f(x, y)$.

Given an equivalence relation E on X and a function $f : E \to \mathbb{R}$, for $x \in X$ denote by $f_x : [x]_E \to \mathbb{R}$ the function $f_x(y) = f(x, y)$.

Definition

Suppose E is a countable Borel equivalence relation. E is *amenable* if there exist positive Borel functions $\lambda^n : E \to \mathbb{R}$ such that

- $\lambda^n_x \in \ell^1([x]_E)$ and $||\lambda^n_x||_1 = 1$,
- $\lim_{n\to\infty} ||\lambda^n_x - \lambda^n_y||_1 = 0$ for $(x, y) \in E$.

Given an equivalence relation E on X and a function $f : E \to \mathbb{R}$, for $x \in X$ denote by $f_x : [x]_E \to \mathbb{R}$ the function $f_x(y) = f(x, y)$.

Definition

Suppose E is a countable Borel equivalence relation. E is *amenable* if there exist positive Borel functions $\lambda^n : E \to \mathbb{R}$ such that

- $\lambda^n_x \in \ell^1([x]_E)$ and $||\lambda^n_x||_1 = 1$,
- $\lim_{n\to\infty} ||\lambda^n_x - \lambda^n_y||_1 = 0$ for $(x, y) \in E$.

Theorem (Connes–Feldman–Weiss, Kechris–Miller)

If μ is any Borel probability measure on X and E is a.e. amenable, then E is a.e. hyperfinite.

Suppose G is a group. A natural action of G on 2^G is given by *left-shifts*:

$$(g \cdot s)(h) = s(g^{-1}h).$$

Suppose G is a group. A natural action of G on 2^G is given by *left-shifts*:

$$(g \cdot s)(h) = s(g^{-1}h).$$

Definition

A subset $S \subseteq 2^G$ is called a *G-subshift* if it is closed in the topology and closed under the above action.

Suppose G is a group. A natural action of G on 2^G is given by *left-shifts*:

$$(g \cdot s)(h) = s(g^{-1}h).$$

Definition

A subset $S \subseteq 2^G$ is called a *G-subshift* if it is closed in the topology and closed under the above action.

Definition

Two G-subshifts $T, S \subseteq 2^G$ are *topologically conjugate* if there exists a homeomorphism $f : S \to T$ which commutes with the left actions.

Definition

A G-subshift S is called *minimal* if it does not contain any proper subshift.

Definition

A G-subshift S is called *minimal* if it does not contain any proper subshift.

Equivalently, a subshift is minimal if every orbit in it is dense.

Definition

A G-subshift S is called *minimal* if it does not contain any proper subshift.

Equivalently, a subshift is minimal if every orbit in it is dense.

Definition

A G-subshift S is *free* if the left action on S is free, i.e. for every $x \in S$: if $g \cdot x = x$, then $g = 1$.

Definition

Recall that a group G is *residually finite* if for every $g \in G$ with $g \neq 1$ there exists a finite-index (normal) subgroup N such that $g \notin N$.

Definition

Recall that a group G is *residually finite* if for every $g \in G$ with $g \neq 1$ there exists a finite-index (normal) subgroup N such that $g \notin N$.

Definition

Given a residually finite group G, the *profinite topology* on G is the one with basis at 1 consisting of finite-index subgroups.

Definition (Toeplitz, Krieger)

A word $x \in 2^G$ is called *Toeplitz* if x is continuous in the profinite topology.

Definition (Toeplitz, Krieger)

A word $x \in 2^G$ is called *Toeplitz* if x is continuous in the profinite topology.

Note

In case $G = \mathbb{Z}$, equivalently a word $x \in 2^{\mathbb{Z}}$ is Toeplitz if for every $k \in \mathbb{Z}$ there exists $p > 0$ such that k *has period* p *in* x, i.e.

$$x(k + ip) = x(k) \quad \text{for all } i \in \mathbb{Z}$$

Definition

A subshift $S \subseteq 2^G$ is Toeplitz if it is generated by a Toeplitz word, i.e. there exists a Toeplitz $x \in 2^G$ such that $S = \mathrm{cl}(G \cdot x)$.

Definition

A subshift $S \subseteq 2^G$ is Toeplitz if it is generated by a Toeplitz word, i.e. there exists a Toeplitz $x \in 2^G$ such that $S = \mathrm{cl}(G \cdot x)$.

Theorem (folklore for $\mathbb{Z}$, Krieger for arbitrary G)

Every Toeplitz subshift is minimal.

It turns out that for any countable group G the topological conjugacy relation of G subshifts is a countable Borel equivalence relation.

It turns out that for any countable group G the topological conjugacy relation of G subshifts is a countable Borel equivalence relation.

Definition

A *block code* is a function

$$\sigma : 2^A \to 2$$

for some finite subset $A \subseteq G$.

It turns out that for any countable group G the topological conjugacy relation of G subshifts is a countable Borel equivalence relation.

Definition

A *block code* is a function

$$\sigma : 2^A \to 2$$

for some finite subset $A \subseteq G$.

A block code induces a G-invariant function $\hat{\sigma} : 2^G \to 2^G$:

$$\hat{\sigma}(x)(g) = \sigma(g^{-1} \cdot x \restriction A).$$

Theorem (Curtis–Hedlund–Lyndon)

Any G-invariant homeomorphism of G-subshifts is given by a block code.

Theorem (Curtis–Hedlund–Lyndon)

Any G-invariant homeomorphism of G-subshifts is given by a block code.

In particular, as there are only countably many block codes, the topological conjugacy relation is a countable Borel equivalence relation.

Question (Gao–Jackson–Seward)

Given a countable group G, what is the complexity of topological conjugacy of **minimal** (or even free minimal) G-subshifts?

Question (Gao–Jackson–Seward)

Given a countable group G, what is the complexity of topological conjugacy of **minimal** (or even free minimal) G-subshifts?

Definition

A Borel equivalence relation E on X is *smooth* if there exists a Borel function $f : X \to \mathbb{R}$ such that

$$x_1 \; E \; x_2 \quad \text{iff} \quad f(x_1) = f(x_2)$$

Question (Gao–Jackson–Seward)

Given a countable group G, what is the complexity of topological conjugacy of **minimal** (or even free minimal) G-subshifts?

Definition

A Borel equivalence relation E on X is *smooth* if there exists a Borel function $f : X \to \mathbb{R}$ such that

$$x_1 \; E \; x_2 \quad \text{iff} \quad f(x_1) = f(x_2)$$

Theorem (Gao–Jackson–Seward)

For any infinite countable group G the topological conjugacy of free minimal G-subshifts is not smooth.

Definition

A group G is *locally finite* if any finitely generated subgroup of G is finite.

Definition

A group G is *locally finite* if any finitely generated subgroup of G is finite.

Theorem (Gao–Jackson–Seward)

If G is locally finite, then the topological conjugacy of free minimal G-subshifts is hyperfinite.

Definition

Note that any countable group G admits a natural *right action* on the set of its free minimal G-subshifts: $S \cdot g = \{x \cdot g : x \in S\}$, where

$$(x \cdot g)(h) = x(hg).$$

Definition

Note that any countable group G admits a natural *right action* on the set of its free minimal G-subshifts: $S \cdot g = \{x \cdot g : x \in S\}$, where

$$(x \cdot g)(h) = x(hg).$$

Note

It is not difficult to see that S and $S \cdot g$ are topologically conjugate for any $g \in G$.

Theorem (S.–Tsankov)

For any residually finite countable groups G that there exists a probability measure μ on the set of free Toeplitz G-subshifts such that

- μ is invariant under the right action of G
- the stabilizers of points in this action are a.e. amenable.

Theorem (folklore)

If a countable group G acts on a probability space preserving the measure and so that

- the induced equivalence relation is amenable,
- a.e. stabilizers are amenable,

then the group G is amenable.

Theorem (folklore)

If a countable group G acts on a probability space preserving the measure and so that

- the induced equivalence relation is amenable,
- a.e. stabilizers are amenable,

then the group G is amenable.

Corollary

For any residually finite non-amenable group G the topological conjugacy relation is not hyperfinite.

Definition

Given a $\mathbb{Z}$-subshift $T \subseteq 2^{\mathbb{Z}}$, its *topological full group* $[[T]]$ consists of all homeomorphisms $f : T \to T$ such that $f(x)$ belongs to the same $\mathbb{Z}$-orbit as x, for all $x \in T$ and there is a continuous function $n : T \to \mathbb{Z}$ such that $f(x) = S^{n(x)}(x)$.

Definition

Given a $\mathbb{Z}$-subshift $T \subseteq 2^{\mathbb{Z}}$, its *topological full group* $[[T]]$ consists of all homeomorphisms $f : T \to T$ such that $f(x)$ belongs to the same $\mathbb{Z}$-orbit as x, for all $x \in T$ and there is a continuous function $n : T \to \mathbb{Z}$ such that $f(x) = S^{n(x)}(x)$.

Theorem (Giordano–Putnam–Skau)

If T, T' are minimal $\mathbb{Z}$-subshifts, then the following are equivalent:

- $[[T]]$ and $[[T']]$ are isomorphic (as groups)
- T is topologically conjugate to T' or to the inverse shift on T'.

Definition

Given a $\mathbb{Z}$-subshift $T \subseteq 2^{\mathbb{Z}}$, its *topological full group* $[[T]]$ consists of all homeomorphisms $f : T \to T$ such that $f(x)$ belongs to the same $\mathbb{Z}$-orbit as x, for all $x \in T$ and there is a continuous function $n : T \to \mathbb{Z}$ such that $f(x) = S^{n(x)}(x)$.

Theorem (Giordano–Putnam–Skau)

If T, T' are minimal $\mathbb{Z}$-subshifts, then the following are equivalent:

- $[[T]]$ and $[[T']]$ are isomorphic (as groups)
- T is topologically conjugate to T' or to the inverse shift on T'.

Theorem (Juschenko–Monod)

If T is a minimal $\mathbb{Z}$-subshift, then $[[T]]$ is amenable.

Definition

Given two Borel equivalence relations E on X and F on Y we say that E is *Borel reducible* to F if there exists a Borel function $f : X \to Y$ such that

$$x_1 \; E \; x_2 \quad \text{iff} \quad f(x_1) \; F \; f(x_2)$$

Definition

Given two Borel equivalence relations E on X and F on Y we say that E is *Borel reducible* to F if there exists a Borel function $f : X \to Y$ such that

$$x_1 \; E \; x_2 \quad \text{iff} \quad f(x_1) \; F \; f(x_2)$$

A countable Borel equivalence relation E is *universal countable Borel* equivalence if any countable Borel equivalence relation is Borel reducible to E.

In terms of Borel-reducibility the two previous theorems show that the topological conjugacy of minimal $\mathbb{Z}$-subshifts is (almost) Borel reducible to the isomorphism of countable amenable groups.

In terms of Borel-reducibility the two previous theorems show that the topological conjugacy of minimal $\mathbb{Z}$-subshifts is (almost) Borel reducible to the isomorphism of countable amenable groups.

Question (Thomas)

What is the complexity of the topological conjugacy of minimal $\mathbb{Z}$-subshifts?

In terms of Borel-reducibility the two previous theorems show that the topological conjugacy of minimal $\mathbb{Z}$-subshifts is (almost) Borel reducible to the isomorphism of countable amenable groups.

Question (Thomas)

What is the complexity of the topological conjugacy of minimal $\mathbb{Z}$-subshifts?

Theorem (Clemens)

The topological conjugacy of (arbitrary, not neccessarily minimal) $\mathbb{Z}$-subshifts is a universal countable Borel equivalence relation.

Note

Recall that a word $x \in 2^{\mathbb{Z}}$ is Toeplitz if for every $k \in \mathbb{Z}$ there exists $p > 0$ such that k *has period* p *in* x, i.e.

$$x(k + ip) = x(k) \quad \text{for all } i \in \mathbb{Z}$$

Note

Recall that a word $x \in 2^{\mathbb{Z}}$ is Toeplitz if for every $k \in \mathbb{Z}$ there exists $p > 0$ such that k *has period* p *in* x, i.e.

$$x(k + ip) = x(k) \quad \text{for all } i \in \mathbb{Z}$$

Notation

Given $x \in 2^{\mathbb{Z}}$ Toeplitz write

$$\mathrm{Per}_p(x) = \{k \in \mathbb{Z} : k \text{ has period } p \text{ in } x\}.$$

Write also

$$H_p(x) = \{0, \ldots, p - 1\} \setminus \mathrm{Per}_p(x).$$

Definition

A Toeplitz word $x \in 2^{\mathbb{Z}}$ is said to have *separated holes* if

$$\lim_{p \to \infty} \min\{|i - j| : i, j \in H_p(x), i \neq j\} = \infty.$$

Definition

A Toeplitz word $x \in 2^{\mathbb{Z}}$ is said to have *separated holes* if

$$\lim_{p\to\infty} \min\{|i-j| : i, j \in H_p(x), i \neq j\} = \infty.$$

Definition

A subshift $S \subseteq 2^{\mathbb{Z}}$ *has separated holes* if it is generated by a Toeplitz word which has separated holes.

Theorem (S.–Tsankov)

The topological conjugacy relation of $\mathbb{Z}$-Toeplitz subshifts with separated holes is amenable.

Theorem (S.–Tsankov)

The topological conjugacy relation of $\mathbb{Z}$-Toeplitz subshifts with separated holes is amenable.

Question

Is it true that the conjugacy relation of all $\mathbb{Z}$-Toeplitz subshifts is hyperfinite?

Theorem (Kaya)

The topological conjugacy relation of $\mathbb{Z}$-Toeplitz subshifts with separated holes is hyperfinite.