Coarse geometry of Polish groups
Lecture 3

Christian Rosendal,
University of Illinois at Chicago

Polish groups and geometry, Paris, June 2016

The coarsely universal locally Roelcke precompact group

Recall that the Roelcke uniformity on a Polish group G is that given by the metric

$$d_\wedge(g,f) = \inf_{h\in G} d(g,h) + d(h^{-1}, f^{-1}),$$

where d is any compatible left-invariant metric on G.

The coarsely universal locally Roelcke precompact group

Recall that the Roelcke uniformity on a Polish group G is that given by the metric

$$d_\wedge(g,f) = \inf_{h\in G} d(g,h) + d(h^{-1}, f^{-1}),$$

where d is any compatible left-invariant metric on G.

For example, the group $\mathbb{G} = \mathrm{Isom}(\mathbb{U})$ of isometries of the Urysohn space $\mathbb{U}$ is a locally Roelcke precompact Polish group, i.e., has a totally $d_\wedge$-bounded identity neighbourhood.

The coarsely universal locally Roelcke precompact group

Recall that the Roelcke uniformity on a Polish group G is that given by the metric

$$d_\wedge(g, f) = \inf_{h \in G} d(g, h) + d(h^{-1}, f^{-1}),$$

where d is any compatible left-invariant metric on G.

For example, the group $\mathbb{G} = \mathrm{Isom}(\mathbb{U})$ of isometries of the Urysohn space $\mathbb{U}$ is a locally Roelcke precompact Polish group, i.e., has a totally $d_\wedge$-bounded identity neighbourhood.

By Zielinski's Theorem, this implies that the completion $\widehat{\mathbb{G}} = \overline{(\mathbb{G}, d_\wedge)}$ is locally compact.

The coarsely universal locally Roelcke precompact group

Recall that the Roelcke uniformity on a Polish group G is that given by the metric

$$d_\wedge(g,f) = \inf_{h\in G} d(g,h) + d(h^{-1}, f^{-1}),$$

where d is any compatible left-invariant metric on G.

For example, the group $\mathbb{G} = \mathrm{Isom}(\mathbb{U})$ of isometries of the Urysohn space $\mathbb{U}$ is a locally Roelcke precompact Polish group, i.e., has a totally $d_\wedge$-bounded identity neighbourhood.

By Zielinski's Theorem, this implies that the completion $\widehat{\mathbb{G}} = \overline{(\mathbb{G}, d_\wedge)}$ is locally compact.

We also showed that $\mathbb{G}$ is isomorphically and coarsely universal, i.e., contains every locally bounded Polish group as a coarsely embedded closed subgroup.

Coarsely proper, modest and cocompact actions

A general fact about the Roelcke uniformity is that the left and right-shift actions of a group G on itself extend to actions on the Roelcke completion

$$\lambda\colon G \curvearrowright \widehat{G} \curvearrowleft G : \rho$$

Coarsely proper, modest and cocompact actions

A general fact about the Roelcke uniformity is that the left and right-shift actions of a group G on itself extend to actions on the Roelcke completion

$$\lambda\colon G \curvearrowright \widehat{G} \curvearrowleft G : \rho$$

Assume also that G is locally Roelcke precompact, whereby $\widehat{G}$ is locally compact.

Then λ and ρ are coarsely proper, i.e., for $K \subseteq \widehat{G}$ compact,

$$\{g \in G \mid \lambda(g)K \cap K \neq \emptyset\}$$

is coarsely bounded (and similarly for ρ).

Then λ and ρ are coarsely proper, i.e., for $K \subseteq \widehat{G}$ compact,

$$\{g \in G \mid \lambda(g)K \cap K \neq \emptyset\}$$

is coarsely bounded (and similarly for ρ).

To see this, let $U \supseteq K$ be open and relatively compact, whence $V = U \cap G$ is Roelcke precompact and thus coarsely bounded.

Then λ and ρ are coarsely proper, i.e., for $K \subseteq \widehat{G}$ compact,

$$\{g \in G \mid \lambda(g)K \cap K \neq \emptyset\}$$

is coarsely bounded (and similarly for ρ).

To see this, let $U \supseteq K$ be open and relatively compact, whence $V = U \cap G$ is Roelcke precompact and thus coarsely bounded.

Then

$$\lambda(g)K \cap K \neq \emptyset$$

Then λ and ρ are coarsely proper, i.e., for $K \subseteq \widehat{G}$ compact,

$$\{g \in G \mid \lambda(g)K \cap K \neq \emptyset\}$$

is coarsely bounded (and similarly for ρ).

To see this, let $U \supseteq K$ be open and relatively compact, whence $V = U \cap G$ is Roelcke precompact and thus coarsely bounded.

Then

$$\lambda(g)K \cap K \neq \emptyset \;\Rightarrow\; gV \cap V \neq \emptyset$$

Then λ and ρ are coarsely proper, i.e., for $K \subseteq \widehat{G}$ compact,

$$\{g \in G \mid \lambda(g)K \cap K \neq \emptyset\}$$

is coarsely bounded (and similarly for ρ).

To see this, let $U \supseteq K$ be open and relatively compact, whence $V = U \cap G$ is Roelcke precompact and thus coarsely bounded.

Then

$$\lambda(g)K \cap K \neq \emptyset \ \Rightarrow \ gV \cap V \neq \emptyset \ \Rightarrow \ g \in VV^{-1}.$$

Then λ and ρ are coarsely proper, i.e., for $K \subseteq \widehat{G}$ compact,

$$\{g \in G \mid \lambda(g)K \cap K \neq \emptyset\}$$

is coarsely bounded (and similarly for ρ).

To see this, let $U \supseteq K$ be open and relatively compact, whence $V = U \cap G$ is Roelcke precompact and thus coarsely bounded.

Then

$$\lambda(g)K \cap K \neq \emptyset \;\Rightarrow\; gV \cap V \neq \emptyset \;\Rightarrow\; g \in VV^{-1}.$$

So $\{g \in G \mid \lambda(g)K \cap K \neq \emptyset\}$ is contained in the coarsely bounded set VV^{-1}.

Moreover, the actions are modest, that is, for $K \subseteq \widehat{G}$ compact and $B \subseteq G$ coarsely bounded,

$$\overline{\lambda(B)K} \text{ is compact.}$$

Moreover, the actions are modest, that is, for $K \subseteq \widehat{G}$ compact and $B \subseteq G$ coarsely bounded,

$$\overline{\lambda(B)K} \text{ is compact.}$$

This simply follows from the fact that Roelcke precompact sets are closed under products and coincides with the coarsely bounded sets.

Moreover, the actions are modest, that is, for $K \subseteq \widehat{G}$ compact and $B \subseteq G$ coarsely bounded,

$$\overline{\lambda(B)K} \text{ is compact.}$$

This simply follows from the fact that Roelcke precompact sets are closed under products and coincides with the coarsely bounded sets.

Note that modesty is automatic when G is actually locally compact, but for example fails for the coarsely proper action

$$S_\infty \curvearrowright \mathbb{N}.$$

Moreover, the actions are modest, that is, for $K \subseteq \widehat{G}$ compact and $B \subseteq G$ coarsely bounded,

$$\overline{\lambda(B)K} \text{ is compact.}$$

This simply follows from the fact that Roelcke precompact sets are closed under products and coincides with the coarsely bounded sets.

Note that modesty is automatic when G is actually locally compact, but for example fails for the coarsely proper action

$$S_\infty \curvearrowright \mathbb{N}.$$

However, only occasionally are the actions cocompact, i.e.,

$$\widehat{G} = \lambda(G)K$$

for some compact $K \subseteq \widehat{G}$.

Groups of bounded geometry

Definition

A Polish group G has bounded geometry if there is a coarsely bounded set B covering every other coarsely bounded set A by finitely many left-translates,

$$A \subseteq FB, \qquad F \subseteq G \text{ finite.}$$

Groups of bounded geometry

Definition

A Polish group G has bounded geometry if there is a coarsely bounded set B covering every other coarsely bounded set A by finitely many left-translates,

$$A \subseteq FB, \qquad F \subseteq G \text{ finite.}$$

A symmetric such set $B \ni 1$ is called a gauge for G, since it gauges the size of A via the Kolmogorov entropy

$$\mathrm{ent}_B(A) = \min\left(|F| \;\middle|\; A \subseteq FB\right).$$

Groups of bounded geometry

Definition

A Polish group G has bounded geometry if there is a coarsely bounded set B covering every other coarsely bounded set A by finitely many left-translates,

$$A \subseteq FB, \qquad F \subseteq G \text{ finite.}$$

A symmetric such set $B \ni 1$ is called a gauge for G, since it gauges the size of A via the Kolmogorov entropy

$$\mathrm{ent}_B(A) = \min\big(|F| \;\big|\; A \subseteq FB\big).$$

Locally compact groups have bounded geometry, since every symmetric identity neighbourhood is a gauge.

Groups of bounded geometry

Definition

A Polish group G has bounded geometry if there is a coarsely bounded set B covering every other coarsely bounded set A by finitely many left-translates,

$$A \subseteq FB, \qquad F \subseteq G \text{ finite.}$$

A symmetric such set $B \ni 1$ is called a gauge for G, since it gauges the size of A via the Kolmogorov entropy

$$\mathrm{ent}_B(A) = \min\left(|F| \;\middle|\; A \subseteq FB\right).$$

Locally compact groups have bounded geometry, since every symmetric identity neighbourhood is a gauge. In fact, these are the only groups with this additional property.

Groups of bounded geometry

Definition

A Polish group G has bounded geometry *if there is a coarsely bounded set B covering every other coarsely bounded set A by finitely many left-translates,*

$$A \subseteq FB, \qquad F \subseteq G \text{ finite.}$$

A symmetric such set $B \ni 1$ is called a gauge for G, since it gauges the size of A via the Kolmogorov entropy

$$\mathrm{ent}_B(A) = \min\big(|F| \;\big|\; A \subseteq FB\big).$$

Locally compact groups have bounded geometry, since every symmetric identity neighbourhood is a gauge. In fact, these are the only groups with this additional property.

But also coarsely bounded groups are gauges for themselves and hence have bounded geometry.

Proposition

Every Polish group of bounded geometry is locally bounded.

Proposition

Every Polish group of bounded geometry is locally bounded.

Indeed, if B is a gauge for G, then either B^4 has non-empty interior and G locally bounded or B^2 is meagre (Pettis' Theorem).

Proposition

Every Polish group of bounded geometry is locally bounded.

Indeed, if B is a gauge for G, then either B^4 has non-empty interior and G locally bounded or B^2 is meagre (Pettis' Theorem).

If B^2 is meagre, then by Mycielski's Independence Theorem there is a Cantor set $C \subseteq G$ so that

$$\forall x \neq y \in C: \quad xB \cap yB = \emptyset.$$

Proposition

Every Polish group of bounded geometry is locally bounded.

Indeed, if B is a gauge for G, then either B^4 has non-empty interior and G locally bounded or B^2 is meagre (Pettis' Theorem).

If B^2 is meagre, then by Mycielski's Independence Theorem there is a Cantor set $C \subseteq G$ so that

$$\forall x \neq y \in C: \quad xB \cap yB = \emptyset.$$

But then C cannot be covered by finitely many left-translates of B, contradicting that C is compact and thus coarsely bounded, while B is a gauge for G.

If $H \curvearrowright X$ is a coarsely proper, modest and cocompact action of a Polish group on a locally compact Hausdorff space, then it is fairly easy to see that H must have bounded geometry.

If $H \curvearrowright X$ is a coarsely proper, modest and cocompact action of a Polish group on a locally compact Hausdorff space, then it is fairly easy to see that H must have bounded geometry.

But what about the converse?

If $H \curvearrowright X$ is a coarsely proper, modest and cocompact action of a Polish group on a locally compact Hausdorff space, then it is fairly easy to see that H must have bounded geometry.

But what about the converse?

Suppose H has bounded geometry, B is a gauge and view H as a coarsely embedded closed subgroup of $\mathbb{G} = \mathrm{Isom}(\mathbb{U})$.

If $H \curvearrowright X$ is a coarsely proper, modest and cocompact action of a Polish group on a locally compact Hausdorff space, then it is fairly easy to see that H must have bounded geometry.

But what about the converse?

Suppose H has bounded geometry, B is a gauge and view H as a coarsely embedded closed subgroup of $\mathbb{G} = \mathrm{Isom}(\mathbb{U})$.

Let X be the closure of H inside the locally compact Roelcke completion $\widehat{\mathbb{G}}$, whence the actions

$$\lambda\colon H \curvearrowright X \curvearrowleft H :\rho$$

are commuting and modest.

If $H \curvearrowright X$ is a coarsely proper, modest and cocompact action of a Polish group on a locally compact Hausdorff space, then it is fairly easy to see that H must have bounded geometry.

But what about the converse?

Suppose H has bounded geometry, B is a gauge and view H as a coarsely embedded closed subgroup of $\mathbb{G} = \mathrm{Isom}(\mathbb{U})$.

Let X be the closure of H inside the locally compact Roelcke completion $\widehat{\mathbb{G}}$, whence the actions

$$\lambda\colon H \curvearrowright X \curvearrowleft H :\rho$$

are commuting and modest.

As H is coarsely embedded in $\mathbb{G}$, the actions are also coarsely proper.

We claim that

$$X = \lambda(H)\overline{B} = \overline{B}\rho(H),$$

whereby the actions are cocompact.

We claim that

$$X = \lambda(H)\overline{B} = \overline{B}\rho(H),$$

whereby the actions are cocompact.

Indeed, suppose $x \in X$.

We claim that

$$X = \lambda(H)\overline{B} = \overline{B}\rho(H),$$

whereby the actions are cocompact.

Indeed, suppose $x \in X$.

As $X = \overline{H} \subseteq \widehat{\mathbb{G}}$, the point x is the $d_\wedge^{\mathbb{G}}$-limit of a sequence (h_i) in H.

We claim that

$$X = \lambda(H)\overline{B} = \overline{B}\rho(H),$$

whereby the actions are cocompact.

Indeed, suppose $x \in X$.

As $X = \overline{H} \subseteq \widehat{\mathbb{G}}$, the point x is the $d_\wedge^{\mathbb{G}}$-limit of a sequence (h_i) in H.

Then $\{h_i\}_i$ is a Roelcke precompact subset of $\mathbb{G}$ and, as H is coarsely embedded in $\mathbb{G}$, the set $\{h_i\}_i$ is coarsely bounded in H.

We claim that

$$X = \lambda(H)\overline{B} = \overline{B}\rho(H),$$

whereby the actions are cocompact.

Indeed, suppose $x \in X$.

As $X = \overline{H} \subseteq \widehat{\mathbb{G}}$, the point x is the $d_\wedge^{\mathbb{G}}$-limit of a sequence (h_i) in H.

Then $\{h_i\}_i$ is a Roelcke precompact subset of $\mathbb{G}$ and, as H is coarsely embedded in $\mathbb{G}$, the set $\{h_i\}_i$ is coarsely bounded in H.

So $\{h_i\}_i \subseteq FB$ for a finite set F and some subsequence (h'_i) is included in a single translate fB.

We claim that

$$X = \lambda(H)\overline{B} = \overline{B}\rho(H),$$

whereby the actions are cocompact.

Indeed, suppose $x \in X$.

As $X = \overline{H} \subseteq \widehat{\mathbb{G}}$, the point x is the $d_\wedge^{\mathbb{G}}$-limit of a sequence (h_i) in H.

Then $\{h_i\}_i$ is a Roelcke precompact subset of $\mathbb{G}$ and, as H is coarsely embedded in $\mathbb{G}$, the set $\{h_i\}_i$ is coarsely bounded in H.

So $\{h_i\}_i \subseteq FB$ for a finite set F and some subsequence (h_i') is included in a single translate fB.

It follows that $x = \lim_i h_i' \in \lambda(f)\overline{B}$ as claimed.

Theorem

The following are equivalent for a Polish group H,

Theorem

The following are equivalent for a Polish group H,

1. *H has bounded geometry,*

Theorem

The following are equivalent for a Polish group H,

1. *H has bounded geometry,*
2. *H is coarsely equivalent to a proper metric space,*

Theorem

The following are equivalent for a Polish group H,

1. *H has bounded geometry,*
2. *H is coarsely equivalent to a proper metric space,*
3. *H admits a continuous, coarsely proper, modest, cocompact action on a locally compact Hausdorff space.*

Theorem

The following are equivalent for a Polish group H,

1. *H has bounded geometry,*
2. *H is coarsely equivalent to a proper metric space,*
3. *H admits a continuous, coarsely proper, modest, cocompact action on a locally compact Hausdorff space.*

Such a group may look very complicated at the local scale, but when viewed from afar it looks like a locally finite graph.

Theorem

The following are equivalent for a Polish group H,

1. *H has bounded geometry,*
2. *H is coarsely equivalent to a proper metric space,*
3. *H admits a continuous, coarsely proper, modest, cocompact action on a locally compact Hausdorff space.*

Such a group may look very complicated at the local scale, but when viewed from afar it looks like a locally finite graph.

Apart from locally compact groups, coarsely bounded groups or various products of these, the main new examples of groups of bounded geometry come via the central extension

$$\mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_{+}(\mathbb{S}^1).$$

Examples (Bounded geometry groups)

- $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$,
- $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q}, <)$,
- $\mathrm{AbsHomeo}_{\mathbb{Z}}(\mathbb{R})$, *i.e., homeos commuting with integral translations so that*

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

(J. Herndon).

Examples (Bounded geometry groups)

- $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$,
- $\text{Aut}_{\mathbb{Z}}(\mathbb{Q}, <)$,
- $\text{AbsHomeo}_{\mathbb{Z}}(\mathbb{R})$, *i.e., homeos commuting with integral translations so that*

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

(J. Herndon).

Note that, for all of these, the tautological action on $\mathbb{R}$ is coarsely proper, modest and cocompact.

Examples (Bounded geometry groups)

- $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$,
- $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q}, <)$,
- $\mathrm{AbsHomeo}_{\mathbb{Z}}(\mathbb{R})$, *i.e., homeos commuting with integral translations so that*

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

(J. Herndon).

Note that, for all of these, the tautological action on $\mathbb{R}$ is coarsely proper, modest and cocompact.

Problem

- *Identify new groups of bounded geometry not originating from semi-direct products of groups hitherto considered.*

Examples (Bounded geometry groups)

- $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$,
- $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q}, <)$,
- $\mathrm{AbsHomeo}_{\mathbb{Z}}(\mathbb{R})$, *i.e., homeos commuting with integral translations so that*

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

(J. Herndon).

Note that, for all of these, the tautological action on $\mathbb{R}$ is coarsely proper, modest and cocompact.

Problem

- *Identify new groups of bounded geometry not originating from semi-direct products of groups hitherto considered.*
- *Is every group of bounded geometry coarsely equivalent to a locally compact group?*

A theorem of M. Gromov

Part of the interest in locally Roelcke precompact groups and groups of bounded geometry is that, in them, we have an informative translation between coarse geometry and topology.

A theorem of M. Gromov

Part of the interest in locally Roelcke precompact groups and groups of bounded geometry is that, in them, we have an informative translation between coarse geometry and topology.

Note that a locally compact Hausdorff space does not have an inherent coarse structure;

A theorem of M. Gromov

Part of the interest in locally Roelcke precompact groups and groups of bounded geometry is that, in them, we have an informative translation between coarse geometry and topology.

Note that a locally compact Hausdorff space does not have an inherent coarse structure; only a notion of $x_n \to \infty$.

A theorem of M. Gromov

Part of the interest in locally Roelcke precompact groups and groups of bounded geometry is that, in them, we have an informative translation between coarse geometry and topology.

Note that a locally compact Hausdorff space does not have an inherent coarse structure; only a notion of $x_n \to \infty$.

Theorem (M. Gromov)

Two countable discrete groups Γ and Λ are coarsely equivalent if and only if they admit commuting, proper, cocompact actions

$$\Gamma \curvearrowright X \curvearrowleft \Lambda$$

on a locally compact Hausdorff space X.

A theorem of M. Gromov

Part of the interest in locally Roelcke precompact groups and groups of bounded geometry is that, in them, we have an informative translation between coarse geometry and topology.

Note that a locally compact Hausdorff space does not have an inherent coarse structure; only a notion of $x_n \to \infty$.

Theorem (M. Gromov)

Two countable discrete groups Γ and Λ are coarsely equivalent if and only if they admit commuting, proper, cocompact actions

$$\Gamma \curvearrowright X \curvearrowleft \Lambda$$

on a locally compact Hausdorff space X.

For Polish groups G and H, a pair $G \curvearrowright X \curvearrowleft H$ of commuting, coarsely proper, modest, cocompact actions is called a topological coupling.

A topological coupling between G and H induces a coarse equivalence.

A topological coupling between G and H induces a coarse equivalence.

But does the converse hold for Polish or even locally compact groups?

A topological coupling between G and H induces a coarse equivalence.

But does the converse hold for Polish or even locally compact groups?

Theorem

Suppose that $\phi\colon H \to G$ is a uniformly continuous coarse equivalence from a locally compact group H to a Polish group G of bounded geometry. Then H and G admit a topological coupling $H \curvearrowright X \curvearrowleft G$.

A topological coupling between G and H induces a coarse equivalence.

But does the converse hold for Polish or even locally compact groups?

Theorem

Suppose that $\phi\colon H \to G$ is a uniformly continuous coarse equivalence from a locally compact group H to a Polish group G of bounded geometry. Then H and G admit a topological coupling $H \curvearrowright X \curvearrowleft G$.

The coupling space X is constructed as follows:

A topological coupling between G and H induces a coarse equivalence.

But does the converse hold for Polish or even locally compact groups?

Theorem

Suppose that $\phi\colon H \to G$ is a uniformly continuous coarse equivalence from a locally compact group H to a Polish group G of bounded geometry. Then H and G admit a topological coupling $H \curvearrowright X \curvearrowleft G$.

The coupling space X is constructed as follows:

- View G as a coarsely embedded closed subgroup of a locally Roelcke precompact group $\mathbb{H}$.

A topological coupling between G and H induces a coarse equivalence.

But does the converse hold for Polish or even locally compact groups?

Theorem

Suppose that $\phi\colon H \to G$ is a uniformly continuous coarse equivalence from a locally compact group H to a Polish group G of bounded geometry. Then H and G admit a topological coupling $H \curvearrowright X \curvearrowleft G$.

The coupling space X is constructed as follows:

- View G as a coarsely embedded closed subgroup of a locally Roelcke precompact group $\mathbb{H}$.
- Let $\overline{G}$ denote the closure of G inside the locally compact Roelcke completion $\widehat{\mathbb{H}}$.

A topological coupling between G and H induces a coarse equivalence.

But does the converse hold for Polish or even locally compact groups?

Theorem

Suppose that $\phi\colon H \to G$ is a uniformly continuous coarse equivalence from a locally compact group H to a Polish group G of bounded geometry. Then H and G admit a topological coupling $H \curvearrowright X \curvearrowleft G$.

The coupling space X is constructed as follows:

- View G as a coarsely embedded closed subgroup of a locally Roelcke precompact group $\mathbb{H}$.
- Let $\overline{G}$ denote the closure of G inside the locally compact Roelcke completion $\widehat{\mathbb{H}}$.
- Then H and G act on $\overline{G}^H$ and we let $X = \overline{H \cdot \phi \cdot G}$.

So we are left with the problem of determining when coarse equivalences can be made uniformly continuous.

So we are left with the problem of determining when coarse equivalences can be made uniformly continuous.

A simple case is when H is totally disconnected. Then a coarse equivalence $\phi\colon H \to G$ can be replaced by one which is constant on left-cosets hV of some compact open subgroup $V \leqslant H$ and hence is uniformly continuous.

Definition

Let G be a Polish group. We say that G is efficiently contractible if there is a contraction $R\colon [0,1] \times G \to G$ with uniformly continuous restrictions

$$R\colon [0,1] \times B \to G,$$

whenever B is coarsely bounded.

Definition

Let G be a Polish group. We say that G is efficiently contractible if there is a contraction $R\colon [0,1] \times G \to G$ with uniformly continuous restrictions

$$R\colon [0,1] \times B \to G,$$

whenever B is coarsely bounded.

For example, contractible Lie groups, Banach spaces or $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Definition

Let G be a Polish group. We say that G is efficiently contractible if there is a contraction $R\colon [0,1]\times G\to G$ with uniformly continuous restrictions

$$R\colon [0,1]\times B\to G,$$

whenever B is coarsely bounded.

For example, contractible Lie groups, Banach spaces or $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Theorem

Suppose $\phi\colon H\to G$ is a bornologous map from a Polish group H of bounded geometry to an efficiently contractible Polish group G.

Definition

Let G be a Polish group. We say that G is efficiently contractible if there is a contraction $R\colon [0,1]\times G\to G$ with uniformly continuous restrictions

$$R\colon [0,1]\times B\to G,$$

whenever B is coarsely bounded.

For example, contractible Lie groups, Banach spaces or $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Theorem

Suppose $\phi\colon H\to G$ is a bornologous map from a Polish group H of bounded geometry to an efficiently contractible Polish group G. Then ϕ is close to a uniformly continuous map $\psi\colon H\to G$, i.e.,

$$\{\phi(h)^{-1}\psi(h)\mid h\in H\} \text{ is coarsely bounded.}$$

Corollary

Let G and H be Polish groups of bounded geometry

Corollary

Let G and H be Polish groups of bounded geometry and assume that either

- *G and H are coarsely equivalent to a totally disconnected locally compact group F,*

Corollary

Let G and H be Polish groups of bounded geometry and assume that either

- *G and H are coarsely equivalent to a totally disconnected locally compact group F, or*
- *G and H are efficiently contractible and coarsely equivalent to a locally compact group F.*

Corollary

Let G and H be Polish groups of bounded geometry and assume that either

- *G and H are coarsely equivalent to a totally disconnected locally compact group F, or*
- *G and H are efficiently contractible and coarsely equivalent to a locally compact group F.*

Then G and H have a topological coupling.

Interplay between the coarse geometry and harmonic analytic or dynamical features

To which extent does the coarse geometry represent a separate aspect or does it interact with other features of Polish groups?

Interplay between the coarse geometry and harmonic analytic or dynamical features

To which extent does the coarse geometry represent a separate aspect or does it interact with other features of Polish groups?

We can look at some distinctive phenomena surrounding Polish groups.

Interplay between the coarse geometry and harmonic analytic or dynamical features

To which extent does the coarse geometry represent a separate aspect or does it interact with other features of Polish groups?

We can look at some distinctive phenomena surrounding Polish groups.

- J. P. R. Christensen and W. Herer discovered the first examples of exotic groups, i.e., Polish groups without unitary representations

Interplay between the coarse geometry and harmonic analytic or dynamical features

To which extent does the coarse geometry represent a separate aspect or does it interact with other features of Polish groups?

We can look at some distinctive phenomena surrounding Polish groups.

- J. P. R. Christensen and W. Herer discovered the first examples of exotic groups, i.e., Polish groups without unitary representations and extremely amenable groups, i.e., all of whose compact flows have a fixed point,

Interplay between the coarse geometry and harmonic analytic or dynamical features

To which extent does the coarse geometry represent a separate aspect or does it interact with other features of Polish groups?

We can look at some distinctive phenomena surrounding Polish groups.

- J. P. R. Christensen and W. Herer discovered the first examples of exotic groups, i.e., Polish groups without unitary representations and extremely amenable groups, i.e., all of whose compact flows have a fixed point,
- M. Megrelishvili showed that there are Polish groups with no non-trivial linear representations on reflexive spaces.

Interplay between the coarse geometry and harmonic analytic or dynamical features

To which extent does the coarse geometry represent a separate aspect or does it interact with other features of Polish groups?

We can look at some distinctive phenomena surrounding Polish groups.

- J. P. R. Christensen and W. Herer discovered the first examples of exotic groups, i.e., Polish groups without unitary representations and extremely amenable groups, i.e., all of whose compact flows have a fixed point,
- M. Megrelishvili showed that there are Polish groups with no non-trivial linear representations on reflexive spaces.

For example, $\mathrm{Homeo}_+([0,1])$ satisfies both.

Interplay between the coarse geometry and harmonic analytic or dynamical features

To which extent does the coarse geometry represent a separate aspect or does it interact with other features of Polish groups?

We can look at some distinctive phenomena surrounding Polish groups.

- J. P. R. Christensen and W. Herer discovered the first examples of exotic groups, i.e., Polish groups without unitary representations and extremely amenable groups, i.e., all of whose compact flows have a fixed point,
- M. Megrelishvili showed that there are Polish groups with no non-trivial linear representations on reflexive spaces.

For example, $\mathrm{Homeo}_+([0,1])$ satisfies both.

As is well known, neither can happen in a locally compact group.

The Ellis–Veech Theorem for Polish groups

R. Ellis and W. Veech has shown that not only are locally compact groups not extremely amenable, but they act freely on a compact Hausdorff space.

The Ellis–Veech Theorem for Polish groups

R. Ellis and W. Veech has shown that not only are locally compact groups not extremely amenable, but they act freely on a compact Hausdorff space.

It turns out that the proof of this has a purely geometric content.

The Ellis–Veech Theorem for Polish groups

R. Ellis and W. Veech has shown that not only are locally compact groups not extremely amenable, but they act freely on a compact Hausdorff space.

It turns out that the proof of this has a purely geometric content.

Proposition

Let G be a Polish group of bounded geometry. Then there is a coarsely bounded set B so that every $g \in G \setminus B$ acts freely on the greatest ambit

$$\mathfrak{A}(G) = \mathrm{spec}(\mathrm{LUC}(G)).$$

The Ellis–Veech Theorem for Polish groups

R. Ellis and W. Veech has shown that not only are locally compact groups not extremely amenable, but they act freely on a compact Hausdorff space.

It turns out that the proof of this has a purely geometric content.

Proposition

Let G be a Polish group of bounded geometry. Then there is a coarsely bounded set B so that every $g \in G \setminus B$ acts freely on the greatest ambit

$$\mathfrak{A}(G) = \mathrm{spec}(\mathrm{LUC}(G)).$$

It follows that all extremely amenable subgroups of G, e.g.,

$$\mathrm{Homeo}_+([0,1]) \leqslant \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) = G,$$

are hidden away inside the coarsely bounded set B.

Of course every locally compact group G has a unitary representation on $L^2(G)$, but does it have an action displaying its coarse geometry?

Of course every locally compact group G has a unitary representation on $L^2(G)$, but does it have an action displaying its coarse geometry?

We are interested in finding coarsely proper continuous isometric actions

$$G \curvearrowright E$$

on Banach spaces E with nice geometry.

Of course every locally compact group G has a unitary representation on $L^2(G)$, but does it have an action displaying its coarse geometry?

We are interested in finding coarsely proper continuous isometric actions

$$G \curvearrowright E$$

on Banach spaces E with nice geometry.

We recall that this means that, for some/every $x \in E$, the orbital map

$$g \in G \mapsto g(x) \in E$$

should be a coarse embedding.

Of course every locally compact group G has a unitary representation on $L^2(G)$, but does it have an action displaying its coarse geometry?

We are interested in finding coarsely proper continuous isometric actions

$$G \curvearrowright E$$

on Banach spaces E with nice geometry.

We recall that this means that, for some/every $x \in E$, the orbital map

$$g \in G \mapsto g(x) \in E$$

should be a coarse embedding.

Observe also that, by the Mazur–Ulam Theorem, such actions are always by affine isometries.

Theorem (Bekka, Chérix, Valette)

Every amenable locally compact group has the Haagerup property, that is, admits a coarsely proper affine isometric action on a Hilbert space.

Theorem (Bekka, Chérix, Valette)

Every amenable locally compact group has the Haagerup property, that is, admits a coarsely proper affine isometric action on a Hilbert space.

Theorem (Brown–Guentner, Haagerup–Przybyszewska)

Every locally compact group admits a coarsely proper affine isometric action on a reflexive space.

Theorem (Bekka, Chérix, Valette)

Every amenable locally compact group has the Haagerup property, that is, admits a coarsely proper affine isometric action on a Hilbert space.

Theorem (Brown–Guentner, Haagerup–Przybyszewska)

Every locally compact group admits a coarsely proper affine isometric action on a reflexive space.

However, in the category of Polish groups the situation is very different.

Theorem (Bekka, Chérix, Valette)

Every amenable locally compact group has the Haagerup property, that is, admits a coarsely proper affine isometric action on a Hilbert space.

Theorem (Brown–Guentner, Haagerup–Przybyszewska)

Every locally compact group admits a coarsely proper affine isometric action on a reflexive space.

However, in the category of Polish groups the situation is very different.

Theorem

$\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ *is a non-Archimedean, amenable, locally Roelcke precompact Polish group and thus admits faithful unitary representations.*

Theorem (Bekka, Chérix, Valette)

Every amenable locally compact group has the Haagerup property, that is, admits a coarsely proper affine isometric action on a Hilbert space.

Theorem (Brown–Guentner, Haagerup–Przybyszewska)

Every locally compact group admits a coarsely proper affine isometric action on a reflexive space.

However, in the category of Polish groups the situation is very different.

Theorem

$\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ *is a non-Archimedean, amenable, locally Roelcke precompact Polish group and thus admits faithful unitary representations. Nevertheless, affine isometric reflexive representations have fixed points.*

Theorem (Bekka, Chérix, Valette)

Every amenable locally compact group has the Haagerup property, that is, admits a coarsely proper affine isometric action on a Hilbert space.

Theorem (Brown–Guentner, Haagerup–Przybyszewska)

Every locally compact group admits a coarsely proper affine isometric action on a reflexive space.

However, in the category of Polish groups the situation is very different.

Theorem

$\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ *is a non-Archimedean, amenable, locally Roelcke precompact Polish group and thus admits faithful unitary representations. Nevertheless, affine isometric reflexive representations have fixed points.*

The last statement here relies on a result of N. Kalton.

Definition

A Polish group G is *Følner amenable* *if either*

- $G = \overline{\bigcup_n K_n}$ *for a sequence of compact subgroups* $K_1 \leqslant K_1 \leqslant \ldots \leqslant G$,
- *there is a locally compact amenable group H and a continuous homomorphism* $\pi \colon H \to G$ *with dense image.*

Definition

A Polish group G is *Følner amenable* *if either*

- $G = \overline{\bigcup_n K_n}$ *for a sequence of compact subgroups* $K_1 \leqslant K_1 \leqslant \ldots \leqslant G$,
- *there is a locally compact amenable group H and a continuous homomorphism $\pi \colon H \to G$ with dense image.*

For example, locally compact amenable groups, Banach spaces, $\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ and $\mathrm{Isom}(\mathbb{U})$.

Definition

A Polish group G is *Følner amenable* *if either*

- $G = \overline{\bigcup_n K_n}$ *for a sequence of compact subgroups* $K_1 \leqslant K_1 \leqslant \ldots \leqslant G$,
- *there is a locally compact amenable group H and a continuous homomorphism $\pi \colon H \to G$ with dense image.*

For example, locally compact amenable groups, Banach spaces, $\mathrm{Isom}(\mathbb{Z}\mathbb{U})$ and $\mathrm{Isom}(\mathbb{U})$.

Theorem

Let G be a Følner amenable Polish group of bounded geometry. Then G admits a coarsely proper affine isometric action on a reflexive space.

Definition

A Polish group G is *Følner amenable* *if either*

- $G = \overline{\bigcup_n K_n}$ *for a sequence of compact subgroups* $K_1 \leqslant K_1 \leqslant \ldots \leqslant G$,
- *there is a locally compact amenable group H and a continuous homomorphism* $\pi \colon H \to G$ *with dense image.*

For example, locally compact amenable groups, Banach spaces, $\mathrm{Isom}(\mathbb{ZU})$ and $\mathrm{Isom}(\mathbb{U})$.

Theorem

Let G be a Følner amenable Polish group of bounded geometry. Then G admits a coarsely proper affine isometric action on a reflexive space.

So again the pathological subgroups with no reflexive representations are hidden away inside a coarsely bounded set in G.