Coarse geometry of Polish groups
Lecture 1

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Polish groups and geometry, Paris, June 2016

A geometric approach to topological groups

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Basic motivating examples include:

- Finitely generated groups and locally compact groups,
- Banach spaces,
- Homeomorphism groups of manifolds.

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From this, we define a *left-invariant* metric on Γ, called the word metric, by

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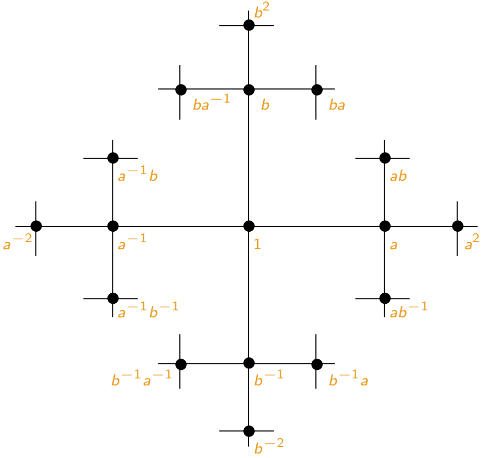
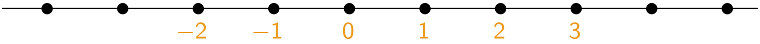
$$\frac{1}{K}\rho_S - C \;\leqslant\; \rho_{S'} \;\leqslant\; K\rho_S + C$$

for some constants K, C.

For example, let $\mathbb{F}_2$ be the free non-abelian group on generators a, b and set $\Sigma = \{1, a, b, a^{-1}, b^{-1}\}$.



Consider first $(\mathbb{Z}, +)$ with generating set $\Sigma_1 = \{-1, 0, 1\}$.



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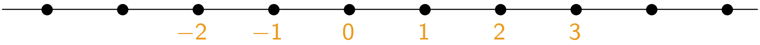

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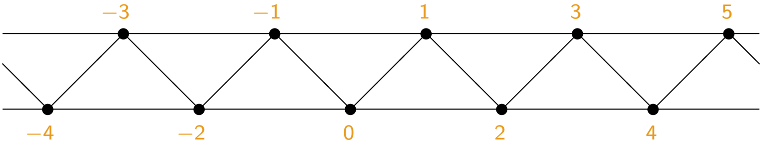

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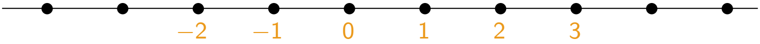

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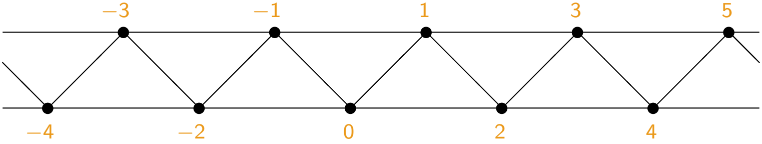

Then

$$\frac{1}{2}\rho_{\Sigma_1} \leqslant \rho_{\Sigma_2} \leqslant \rho_{\Sigma_1}.$$

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By the Baire category theorem, some power K^p has non-empty interior, so if K_1, K_2 are two such sets, then

$$K_1 \subseteq K_2^n, \quad \text{and} \quad K_2 \subseteq K_1^m$$

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Therefore,

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So, up to quasi-isometry, ρ_K is independent of K.

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Any two such metrics d and d' will be coarsely equivalent, that is,

$$\kappa\big(d(x,y)\big) \leqslant d'(x,y) \leqslant \omega\big(d(x,y)\big)$$

for functions $\kappa, \omega \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \kappa(t) = \infty$.

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Observe that this is weaker than being quasi-isometric.

Example 3: Banach spaces

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Geometric non-linear functional analysis
$=$ *Geometric group theory of Banach spaces*

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A uniform space is intended to capture the idea of being uniformly close in a topological space and hence gives rise to concepts of Cauchy sequences and completeness.

Pseudometric spaces

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Recall here that an écart on X is a map $d\colon X \times X \to \mathbb{R}_+$ satisfying

- $d(x, x) = 0$,
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and define a uniformity $\mathcal{U}_d$ by

$$\mathcal{U}_d = \{E \subseteq X \times X \mid \exists \alpha > 0 \;\; E_\alpha \subseteq E\}.$$

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Morphisms

Recall that a map $\phi\colon (X,\mathcal{U}) \to (M,\mathcal{V})$ between uniform spaces is uniformly continuous if

$$\forall F \in \mathcal{V}\ \ \exists E \in \mathcal{U}\colon \quad (x,y) \in E \ \ \Rightarrow \ \ \big(\phi(x),\phi(y)\big) \in F.$$

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E.g., a map $\phi\colon (X,d) \to (M,\partial)$ is bornologous if

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Isomorphisms

A coarse embedding $\phi\colon (X, d) \to (M, \partial)$ is a coarse equivalence if, moreover, the image is cobounded, that is,

$$\sup_{z \in M} \partial(z, \phi[X]) < \infty.$$

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Namely, a coarse equivalence is a pair of bornologous maps

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$$\sup_{x\in X} d\big(\psi\phi(x),x\big) < \infty \quad \& \quad \sup_{z\in M} \partial\big(\phi\psi(z),z\big) < \infty.$$

Left-uniform structure on a topological group

If G is a topological group, its left-uniformity $\mathcal{U}_L$ is that generated by entourages of the form

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A basic theorem, due essentially to G. Birkhoff (fils) and S. Kakutani, is that

$$\mathcal{U}_L = \bigcup_d \mathcal{U}_d,$$

where the union is taken over all continuous left-invariant écarts d on G, i.e., so that

$$d(zx, zy) = d(x, y).$$

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If G is a topological group, its left-coarse structure $\mathcal{E}_L$ is given by

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One may easily show that the class of coarsely bounded subsets is an ideal of subsets of G stable under the operations

$$A \mapsto A^{-1}, \quad (A, B) \mapsto AB \quad \text{and} \quad A \mapsto \overline{A}.$$

Proposition

The left-coarse structure $\mathcal{E}_L$ on a topological group G is generated by entourages of the form

$$E_A = \{(x, y) \mid x^{-1}y \in A\},$$

where A varies over coarsely bounded sets.

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A subset A of a Polish group G is coarsely bounded if and only if, for every identity neighbourhood V, there are a finite set $F \subseteq G$ and $k \geqslant 1$ so that

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- Similarly, in the underlying additive group $(X, +)$ of a Banach space $(X, \|\cdot\|)$, they are the norm bounded subsets.

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In case d is a compatible left-invariant écart inducing the coarse structure on G, that is, $\mathcal{E}_L = \mathcal{E}_d$, we say that d is coarsely proper.

Thus, d is coarsely proper if and only if the finite d-diameter subsets of G are simply the coarsely bounded sets.

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If d and d' are both coarsely proper metrics on G, then $\mathcal{E}_d = \mathcal{E}_L = \mathcal{E}_{d'}$, so d are d' are coarsely equivalent, i.e.,

$$\kappa(d(x,y)) \leqslant d'(x,y) \leqslant \omega(d(x,y))$$

for some functions κ, ω as before.

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Indeed, in a countable product $\prod_n G_n$, the coarsely bounded sets are contained in products $\prod_n B_n$ of coarsely bounded sets $B_n \subseteq G_n$.

Coarse versus quasi-metric structure

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So one can identify the quasi-metric structure as that identified by any such word metric.

Moreover, a coarse equivalence between such groups is always a quasi-isometry, so you may think of quasi-isometry in place of coarse equivalence throughout.

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Suppose (X, d) is a separable complete metric space and let $\mathrm{Isom}(X, d)$ denote its group of isometries equipped with the Polish topology of pointwise convergence; i.e.,

$$g_i \to g \quad \Leftrightarrow \quad g_i(x) \to g(x) \text{ for all } x \in X.$$

Note that, for any fixed $x_0 \in X$, the formula

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So, for any choice of $x_0 \in \mathbb{U}$,

$$g \in \mathrm{Isom}(\mathbb{U}) \mapsto g(x_0) \in \mathbb{U}$$

is a coarse equivalence between $\mathrm{Isom}(\mathbb{U})$ and $\mathbb{U}$.

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This includes automorphism groups of countable $\aleph_0$-categorical structures, such as

$$S_\infty, \quad \mathrm{Aut}(\mathbb{Q}, <), \quad \mathrm{Homeo}(\{0,1\}^{\mathbb{N}}), \quad \mathrm{Aut}(\mathcal{R}),$$

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