Metric ultraproducts of metric groups

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Main results

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More precisely, there is a countable sequence of finite normed groups such that any finite subset of any normed group is ε-approximated by all but finitely many of them, for any $\varepsilon > 0$.

Theorem

There is a sequence of finite normed groups whose metric ultraproduct contains isometrically as a subgroup every separable normed topological group.

A metric d on a group G is *left-invariant* if $d(x, y) = d(g \cdot x, g \cdot y)$ for every $x, y, g \in G$. Right-invariance and bi-invariance are defined analogously.

Basic notions

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A norm (or length function/value) on a group G is a function $\lambda : G \to \mathbb{R}_0^+$ satisfying:

1. $\lambda(g) = 0$ iff $g = 1_G$;
2. $\lambda(g) = \lambda(g^{-1})$, for every $g \in G$;
3. $\lambda(g \cdot h) \leq \lambda(g) + \lambda(h)$, for every $g, h \in G$.

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If a norm λ additionally satisfies $\lambda(g^{-1} \cdot h \cdot g) = \lambda(h)$, then we shall call it *conjugacy-invariant*.

Basic notions

If λ is a norm on G, then

- the formula $d^L_\lambda(g,h) = \lambda(g^{-1} \cdot h)$ defines a left-invariant metric,
- the formula $d^R_\lambda(g,h) = \lambda(h \cdot g^{-1})$ defines a right-invariant metric.

If λ was conjugacy-invariant, then $d^L_\lambda = d^R_\lambda$ is a bi-invariant metric.

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If λ was conjugacy-invariant, then $d^L_\lambda = d^R_\lambda$ is a bi-invariant metric.

Conversely, if d is left- or right-invariant metric, then the formula $\lambda_d(g) = d(g, 1_G)$ defines a norm.
Moreover, if d was bi-invariant, then λ_d is conjugacy-invariant.

Continuity of group operations

Let G be a group with a norm λ. Then G with the topology given by λ is a topological group iff

for every $g \in G$ for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\forall h \in G \ (\lambda(h) < \delta \Rightarrow \lambda(g^{-1} \cdot h \cdot g) < \varepsilon).$$

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We shall such norms *continuous*.

Example: Conjugacy-invariant norms.

Non-example: Take F_∞ freely generated by $(g_n)_{n\in\mathbb{N}}$. For any reduced word $w_1 \ldots w_m$ over the alphabet $\{g_i, g_i^{-1} : i \in \mathbb{N}\}$ set

$$\lambda(w_1 \ldots w_m) = \sum_{i=1}^{m} \rho(w_i)$$

where $\rho(w_i) = 1/n$ iff $w_i = g_n$ or $w_i = g_n^{-1}$.

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where $\rho(w_i)=1/n$ iff $w_i=g_n$ or $w_i=g_n^{-1}$.
Then $\lim_i \lambda(g_i)=0$, however $\lim_i \lambda(g_1^{-1}\cdot g_i\cdot g_1)=2$.

Birkhoff-Kakutani

Theorem [Birkhoff-Kakutani]

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Fact

For G a Hausdorff topological group, TFAE:

- G is SIN, i.e. it admits a neighborhood basis of the identity consisting of open sets closed under conjugation,
- G admits a compatible norm λ such that there is a single modulus of continuity for every $g \in G$ for the function $h \to \lambda(g^{-1} \cdot h \cdot g)$,
- G admits a conjugacy-invariant norm.

Let $\mathcal{C}$ be a class of normed groups (with norms bounded by 1). Say that a (discrete) group G is $\mathcal{C}$-approximable if there is some constant $K \in (0,1)$ such that for any finite subset $F \subseteq G$ and any $\varepsilon > 0$ there is $(H, \lambda) \in \mathcal{C}$ and a map $\phi : F \to H$ such that

Metric approximation of discrete groups

Let $\mathcal{C}$ be a class of normed groups (with norms bounded by 1). Say that a (discrete) group G is $\mathcal{C}$-approximable if there is some constant $K \in (0,1)$ such that for any finite subset $F \subseteq G$ and any $\varepsilon > 0$ there is $(H, \lambda) \in \mathcal{C}$ and a map $\phi : F \to H$ such that

1. $\lambda(\phi(1_G)) < \varepsilon$, if $1_G \in F$;
2. $d^L_\lambda(\phi(g \cdot h), \phi(g) \cdot \phi(h)) < \varepsilon$, for all $g, h \in F$;
3. $d^L_\lambda(\phi(g), \phi(h)) > K$, for all $g, h \in F$.

Metric approximation of discrete groups

Examples:

- Sofic groups: groups approximable by finite permutation groups equipped with the normalized Hamming norm; i.e for $\rho \in S_N$, $\lambda_H(\rho) = 1/n \cdot |\{i \leq n : \rho(i) \neq i\}|$.

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- Hyperlinear groups: groups approximable by groups of finite-dimensional unitary matrices equipped with the normalized 'Hilbert-Schmidt' norm; for $u \in U(n)$, set $\lambda_{HS}(u) = 1/\sqrt{n} \cdot \|\mathrm{Id} - u\|_{HS}$.

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- Weakly sofic group: groups approximable by finite groups equipped with an arbitrary conjugacy-invariant norm.
- Weakly hyperlinear groups: groups approximable by compact groups with an arbitrary conjugacy-invariant norm.

Metric ultraproducts

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $(G_n)_n$ be a sequence of some metric structures, uniformly metrically bounded, and with equi-uniformly continuous operations.

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Then $d_{\mathcal{U}}((g_n)_n, (h_n)_n) = \lim_{\mathcal{U}} d(g_n, h_n)$, for $(g_n)_n, (h_n)_n \in \prod_n G_n$ is a complete pseudometric on $\prod_n G_n$ and the equivalence relation given by having pseudodistance zero is a congruence relation with respect to the operations. So we can form a quotient denoted by $\prod_{\mathcal{U}} G_n$ called the metric ultraproduct of the sequence $(G_n)_n$.

Metric ultraproducts

Example: Let $(G_n, \lambda_n)_n$ be a sequence of groups with conjugacy-invariant norms bounded by 1.

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Then $\prod_{\mathcal{U}} G_n$ is a group with a conjugacy-invariant norm.It is analogously defined as the metric quotient $(G_n)_{\ell_\infty}/\mathcal{N}$, where $(G_n)_{\ell_\infty}$ is $\prod_n G_n$ with the supremum norm and $\mathcal{N} = \{(g_n)_n \in \prod_n G_n : \lim_{\mathcal{U}} \lambda_n(g_n) = 0\}$ is the set of sequences of elements whose norms converge to zero along $\mathcal{U}$. That is easily checked to be a closed normal subgroup.

Embeddings into metric ultraproducts

Let again $\mathcal{C}$ be some class of groups equipped with conjugacy-invariant norm (bounded by 1).

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Fact

A countable group G is $\mathcal{C}$-approximable iff there are a sequence $(G_n)_n \subseteq \mathcal{C}$, a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and an algebraic monomorphism $\phi : G \hookrightarrow \prod_{\mathcal{U}} G_n$ (where the elements of $\phi[G]$ are uniformly separated).

Weakly-sofic groups

Theorem

There is a universal separable group with conjugacy-invariant norm bounded by 1, i.e. every separable group with conjugacy-invariant norm bounded by 1 embeds into it via an isometric homomorphism.

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The group is a completion of a direct limit of finitely generated free groups with distinguished finite subsets and conjugacy-invariant norms, bounded by 1, on them.
That is, we have a sequence $(F_1, A_1, \lambda_1) \leq (F_2, A_2, \lambda_2) \leq \ldots (F_m, A_m, \lambda_2) \leq \ldots$, where F_n is a free of n free generators and $A_n \subseteq F_n$ a finite subsets, λ_n a 'partial' conjugacy-invariant norm on A_n.

Weakly-sofic groups

Consequence

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Groups with conjugacy-invariant norms bounded by 1 are approximable by finitely generated free groups with 'finitely-determined' norm. Actually, the same is true without the boundedness condition.

Problem

Finitely generated free groups are residually finite. Can one use that along with the constructions above to prove that every group is weakly sofic, resp. a stronger assertion that every group with conjugacy-invariant norm is approximable by finite groups with conjugacy-invariant norm?

Let (G, λ) be a normed group. For every $g \in G$ define $\Gamma_g : [0, \infty) \to [0, \infty)$ the *minimal modulus of continuity* as follows:

Approximation by finite groups

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$$\Gamma_g(r) = \sup\{\lambda(g^{-1} \cdot h \cdot g), \lambda(g \cdot h \cdot g^{-1}) : \lambda(h) \leq r\}.$$

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λ on G is conjugacy-invariant if and only if for every $g \in G$, $\Gamma_g = \mathrm{id}$, i.e. $\Gamma_g(r) = r$ for every $g \in G$ and $r \in [0, \sup_{h \in G} \lambda(h)]$.

Proposition

Let (F, λ) be a finitely generated free group with a conjugacy-invariant norm. Let $A \subseteq F$ be a finite subset and let $\varepsilon > 0$. Then there exist a finite normed group (H, ρ) and a map $\phi : A \to H$ such that

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1. the map ϕ is a partial monomorphism preserving the norm (resp. isometric);

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1. the map ϕ is a partial monomorphism preserving the norm (resp. isometric);
2. for every $a \in A$, we have $\Gamma_{\phi(a)} \leq (1 + \varepsilon)\mathrm{id}$, i.e. for every $a \in A$ and $r \in [0, \max_{h \in H} \rho(h)]$ we have

$$\Gamma_{\phi(a)}(r) \leq r + \varepsilon r.$$

Main theorem

Theorem

There exists a countable sequence of finite normed groups (G_n, λ_n) such that for every normed topological group (H, ρ) and any finite subset $A \subseteq H$ and ε there exists N_0 such that for every $n \geq N_0$ there exists a map $\phi : A \to G_n$ such that for every $a \in A$

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- $$|\rho(a) - \lambda_n(\phi(a))| < \varepsilon,$$

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- $$|\rho(a) - \lambda_n(\phi(a))| < \varepsilon,$$
- $$\Gamma_a^H \approx \Gamma_{\phi(a)}^{G_n}.$$

Proof ideas

Observation

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Every finitely generated normed group (G, ρ) may be viewed as a finitely generated free group with a pseudonorm.

Let $g_1, \ldots, g_n$ be some generators of G. Let F be a free group freely generated by $g_1, \ldots, g_n$ and let $\phi : F \hookrightarrow G$ be the canonical epimorphism. Then we define a pseudonorm λ on F by setting

$$\lambda(f) = \rho(\phi(f)),$$

for every $f \in F$.

Proof ideas

Lemma

Let F be a finitely generated free group with a pseudonorm λ. Let $A \subseteq F$ be a finite subset and $\varepsilon > 0$. Then there exists a rational norm ρ on F (determined by values on A) such that for every $a \in A$,

$$|\lambda(a) - \rho(a)| < \varepsilon.$$

Proof ideas

Lemma

Let $F_1, \ldots, F_n$ be finitely generated free groups with norms $\lambda_1, \ldots, \lambda_n$. Then there exists a norm λ on $F_1 * \ldots * F_n$ such that

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- for every $i \leq n$, $\lambda \restriction F_i = \lambda_i$,
- for every free generator $g \in F_i$, $i \leq n$, $\Gamma_g^{F_i} \approx \Gamma_g^{F_1 * \ldots * F_n}$.

Proof ideas

Proposition

Let (F, λ) be a finitely generated free group with a norm. Let $A \subseteq F$ be a finite subset. Then there exist a finite normed group (H, ρ) and a map $\phi : A \to H$ such that

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Proposition

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1. the map ϕ is a partial monomorphism preserving the norm;
2. for each $a \in A$, $\Gamma^F_a \approx \Gamma^H_{\phi(a)}$.

Metric ultraproducts

Let $(G_n, \lambda_n)_n$ be a sequence of normed groups and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $(G_n)_{\ell_\infty}$ be the restricted direct product $\{(g_n)_n \in \prod_n G_n : \sup_n \lambda_n(g_n) < \infty\}$. Let $(G_n)_{\mathcal{C}}$ be the subset of elements "continuous" in the ultraproduct, i.e. the set

$$\{(g_n)_n \in (G_n)_{\ell_\infty} : \forall \varepsilon > 0 \; \exists \delta > 0 \; \exists A \in \mathcal{U}$$

$$\forall n \in A \; \forall h \in G_n (\lambda_n(h) < \delta \Rightarrow \lambda_n(g_n^{-1} \cdot h \cdot g_n) < \varepsilon \wedge \lambda_n(g_n \cdot h \cdot g_n^{-1}) < \varepsilon\}.$$

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$(G_n)_{\mathcal{C}}$ is a subgroup of $(G_n)_{\ell_\infty}$. If all the norms λ_n were conjugacy-invariant (or equicontinuous) then $(G_n)_{\mathcal{C}} = (G_n)_{\ell_\infty}$.

If we define the ultraproduct pseudonorm $\lambda_{\mathcal{U}}$ on $(G_n)_{\mathcal{C}}$, i.e. we set

$$\lambda_{\mathcal{U}}((g_n)_n) = \lim_{\mathcal{U}} \lambda_n(g_n)$$

for every $(g_n)_n \in (G_n)_{\mathcal{C}}$, then $\lambda_{\mathcal{U}}$ is continuous. Therefore, the set of $\lambda_{\mathcal{U}}$-zero elements $N = \{(g_n)_n : \lambda_{\mathcal{U}}((g_n)_n) = 0\}$ is a closed normal subgroup.
We denote the quotient $(G_n)_{\mathcal{C}}/N$ by $(G_n)_{\mathcal{U}}$.

Metric ultraproducts

Facts

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- If all the (G_n, λ_n)'s are conjugacy-invariant, then $(G_n)_{\mathcal{U}}$ is the standard ultraproduct.
- $(G_n)_{\mathcal{U}}$ is complete, i.e. whenever $(u_n)_n \subseteq (G_n)_{\mathcal{U}}$ is a sequence from the ultraproduct such that both $(u_n)_n$ and $(u_n^{-1})_n$ are Cauchy, then there is $u \in (G_n)_{\mathcal{U}}$ such that

$$\lim_n u_n = u$$

and

$$\lim_n u_n^{-1} = u^{-1}.$$

Metric ultraproducts

Facts

- There exists a sequence of finite normed groups $(G_n, \lambda_n)_n$ such that for any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we have that $(G_n)_{\mathcal{U}} = \{1\}$.

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- For every normed group (G, λ) and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we have $G \hookrightarrow_{\mathrm{iso}} (G)_{\mathcal{U}}$.

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- For every normed group (G, λ) and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we have $G \hookrightarrow_{\text{iso}} (G)_\mathcal{U}$.

  However, for example consider S_∞ with some compatible norm, e.g.

  $$\lambda_1(\rho) = 1/\min\{n : \rho(n) \neq n\}$$

  or

  $$\lambda_2(\rho) = \sum_{i:\rho(i)\neq i} 1/2^i.$$

  Then for any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we have

  $$(S_\infty)_\mathcal{U} \cong S_\infty.$$

Metric ultraproducts

Theorem

Let (G_n, λ_n) be the sequence of the finite normed groups from the main theorem and $\mathcal{U}$ be some non-principal ultrafilter on $\mathbb{N}$. Then $(G_n)_\mathcal{U}$ contains isometrically as a subgroup every separable normed topological group.

Let $\mathbb{H}$ be an arbitrary separable normed topological group. Let $H \subseteq \mathbb{H}$ be a countable dense subgroup.

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Choose an increasing sequence of finite subsets $H_1 \subseteq H_2 \subseteq \ldots$ such that $H = \bigcup_n H_n$, and a decreasing sequence $(\varepsilon_n)_n$ such that $\lim_n \varepsilon_n = 0$.

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By the main theorem, there is an increasing sequence $1 \leq k_1 < k_2 < \ldots$ such that for every n and every $k_n \leq i < k_{n+1}$ there is a map $\phi_i : H_n \to G_i$ which is preserves the norm by an ε_n error and does not change the moduli of continuity "too much".

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Then for every $h \in H$, for all but finitely many $i \in \mathbb{N}$, $\phi_i(h)$ is defined. Thus we define $\Phi : H \hookrightarrow (G_n)_{\mathcal{U}}$ by

$$\Phi(h) = (\phi_i(h))_i.$$

Metric ultraproducts

It follows that $\Phi : H \hookrightarrow (G_n)_{\mathcal{U}}$ is an isometric embedding for non-principal ultrafilter (for any ultrafilter extending the filter of co-finite sets).

Since $(G_n)_{\mathcal{U}}$ is complete, it contains isometrically $\mathbb{H}$.

Problems

Let (F, λ) be a finitely generated free group with a conjugacy-invariant norm, $A \subseteq F$ a finite subset and $\varepsilon > 0$. Does there exist a finite group (H, ρ) with a conjugacy-invariant norm and a map $\phi : A \to H$ such that

- ϕ is an ε-approximate homomorphism, i.e. $d_\rho(\phi(a \cdot b), \phi(a) \cdot \phi(b)) < \varepsilon$ for every $a, b \in A$,

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- ϕ is an ε-approximate homomorphism, i.e. $d_\rho(\phi(a \cdot b), \phi(a) \cdot \phi(b)) < \varepsilon$ for every $a, b \in A$,
- $|\lambda(a) - \rho(\phi(a))| < \varepsilon$, for every $a \in A$?